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Cumulative Prospect Theory and Mean Variance Analysis. A Rigorous Comparison

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Cumulative prospect theory and mean variance analysis. A rigorous comparison*

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Abstract

We compare asset allocations derived for cumulative prospect theory (CPT) based on two different methods: Maximizing CPT along the mean–variance efficient frontier and maximizing it without that restriction. We find that with normally distributed returns the difference is negligible. However, using standard asset allocation data of pension funds the difference is considerable. Moreover, with derivatives like call options the restriction to the mean-variance efficient frontier results in a sizable loss of e.g. expected return and expected utility.

Keywords: Cumulative Prospect Theory, Mean–Variance Analysis.
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1 Introduction

Ever since Markowitz (1952) [24], the mean–variance model has been the main paradigm for asset allocations in finance. Its solutions are well understood and efficient algorithms computing them are easily available. However, during the behavioral revolution in economics Kahneman and Tversky (1979) [19] have amassed evidence that actual investors’ decisions depart from the mean–variance model in many ways. Investors consider the deviations of their terminal wealth from a reference level as gains and losses and react differently to gains than to losses. Furthermore, they systematically distort probabilities. This raises the question how asset allocations can be determined for prospect theory. Since the prospect theory (PT) utility function is not differentiable and s–shaped and since the function describing probability distortions is also s–shaped, computing prospect theory asset allocations is non–trivial. Moreover original prospect theory violates first–order stochastic dominance; Tversky and Kahneman (1992) [33] resolved this difficulty by developing cumulative prospect theory (CPT).

Due to the non–differentiability and non–concavity of the components of prospect theory, analytic solutions for prospect theory asset allocations can only be expected in very simple cases. Thus to answer the question how asset allocations based on prospect theory look like, one has to follow a computational approach. To do so, Levy and Levy (2004) [23] suggested a simple solution: Restrict your attention to mean–variance optimal portfolios and then select that portfolio that has the highest prospect utility. As Levy and Levy (2004) [23] have shown this procedure works perfectly fine for normally distributed returns. In that case nothing is lost by restricting to the mean–variance efficient frontier. In the framework of a risk–free and several risky assets, Pirvu and Schulze (2012) [25] generalize the result of Levy and Levy for the class of elliptically symmetric distributions of the risky assets. Allowing for unlimited short–sales, the authors also present an analytical solution which is essentially equivalent to maximizing the CPT objective function along the MV–frontier. Yet, many asset allocation problems involve non–elliptically distributed returns since even on the level of broad indices, returns of stocks, bonds and commodities typically have fat tails and are skewed. Moreover, today many investors include derivatives into their portfolio so that the assumption of elliptically distributed returns is not reasonable at all.

For general return distributions the following analytical results are known. Concerning the theoretically interesting special case of portfolios consist-

ing of two assets, one of them being the risk-free asset, several authors have achieved interesting analytical results. Without probability distortion, Gomes (2005) [13] presents explicit formulas for the optimal asset allocation for a two-state discrete distribution of the risky asset and employs numerical techniques for a general distribution of that asset. For the CPT-case, i.e., with probability distortion Bernard and Ghossoub (2010) [4] as well as He and Zhou (2011) [14] have recently derived explicit formulas for the optimal portfolios and presented a detailed analysis of them. In the above approaches the piecewise-power value function is considered, as suggested by Kahneman and Tversky in their paper (1979) [19] concerning the (original) prospect theory (PT). Regarding explicit analytical solutions, such solutions have been provided also for the piecewise-quadratic value function by Hens and Bachmann (2008) [15] for PT and by Zakamouline and Koekebakker (2009) [35] for CPT. Except for Bernard and Ghossoub (2010) [4] where short-sale is forbidden for the risky asset, the analysis in the other papers is carried out allowing unrestricted short-sales.

While those results have been obtained for the single period model, for the general case involving several risky assets important analytical results have been achieved in the continuous-time portfolio selection framework, involving complete markets and Itô processes for the prices of the risky assets. For the PT case Berkelaar, Kouwenberg and Post (2004) [3] present closed-form solutions for the optimal portfolios. For the general CPT case involving probability distortion, Jin and Zhou (2008) [18] derive general analytical results concerning the optimal investment strategy. Reichlin (2012) [27] proposes numerically tractable methods for computing the optimal CPT portfolio, by analyzing stability properties of the behavioral portfolio selection problem.

Moreover, an interesting analytical result is obtained in Barberis and Huang (2008) [2]. In an asset-pricing framework the authors explore the effect of skewness on the optimal portfolio choice. They assume a normal distribution of the risky assets and append these assets by a small, independent and positively skewed security. The authors find that unlike mean-variance investors, CPT-investors have a definitive preference for positively skewed securities, which they attribute to probability weighting. For the two-assets case, this effect has been analyzed also by Bernard and Ghossoub (2010) [4].

While Levy and Levy (2004) [23] and Pirvu and Schulze (2012) [25] allow for unlimited short sales, we assume short sale restrictions. As the case of normally distributed return shows, this difference in the model formulation does not create the observed differences between CPT and MV. Moreover,

this restriction makes our results closer to applications, e.g. for unsophisticated private investors and for pension funds.

The main purposes of our paper are the following.

We propose a numerical optimization approach for solving portfolio selection problems with several assets, involving objective functions from the cumulative prospect theory. Based on our implementation of the suggested algorithm, we present the results of a numerical study with the following goal. Based on a real-life data-set for asset returns and a generally-used data-set concerning prospect theory parameters including probability distortions, we test numerically the following theoretical results, obtained by different authors under various assumptions concerning the distribution of the asset returns:

- The asset allocations obtained by solving the portfolio optimization problem with a (C)PT objective are located along the mean–variance efficient frontier. Thus maximizing a (C)PT objective along the efficient MV–frontier provides a good approximation to the solution of the portfolio selection problem with a (C)PT objective.
- Asset allocations obtained via the original prospect theory and those resulting from cumulative prospect theory differ substantially.
- Investors with a (C)PT objective function prefer asset allocations with a high skewness (skewness–loving preferences).

We compare the optimal portfolios obtained for the PT and CPT–optimization with those from maximizing a PT/CPT objective function along the mean–variance frontier of Markowitz (1952) [24]. The computations have been carried out by utilizing a monthly–returns benchmark data-set with 8 indices representing asset classes; we would like to express our thanks to Dieter Niggeler of BhFS¹ for providing us with these data. This data-set is standard in computing strategic asset allocations, e.g. for pension funds or private investors.

In this paper we work with the piecewise–power value function of Kahneman and Tversky (1979) [19] throughout. Concerning the parameters of this function, instead of varying them in an artificial fashion, we choose the parameters of the 48 subjects obtained via parameter–free measurement by

¹BhFS stands for Behavioral Finance Solutions, which is a spin-off company of the University of Zurich, see www.bhfs.ch

Abdellaoui, Bleichrodt and Paraschiv (2007) [1], see Appendix C in that paper, which also includes measurement results concerning probability distortion. Our numerical tests have been carried out by taking into account probability distortion.

For comparing the optimal portfolios obtained by (C)PT-optimization with those resulting by maximizing the (C)PT objective function along the MV efficient frontier, we employ three generally used indices from the literature, based on objective function values and on differences in the certainty equivalents. In addition, we also compute the distance of the optimal (C)PT portfolios to the MV-frontier.

The computations have been first carried out by utilizing an empirical distribution computed via k -means clustering from the historical data. Next we added a call option to the empirical distribution data, in order to test the effects of larger deviations from the normal distribution assumption and to test the skewness-loving CPT-investor attitude. Finally, we have worked with a test data-set generated according to a multivariate normal distribution with the same expected value vector and covariance matrix as the original empirical distribution. Via sampling and subsequent clustering an additional empirical distribution has been computed for this purpose.

Our main contributions can be summarized as follows:

- We propose and numerically test a general algorithm for CPT-optimization.
- Our numerical results concerning the original data-set indicate that the CPT-optimization approach results in optimal portfolios which differ for a substantial number of our investors from the results obtained by optimization along the frontier. This difference increases considerably for the data-set involving the call option. For the data-set involving the normal distribution the difference diminishes to a large extent.
- We have also found that portfolios obtained from PT- optimization and those from CPT-optimization differ substantially.
- Finally our numerical tests provide evidence concerning the skewness-loving attitude of (C)PT investors.
- All of our numerical tests have been carried out without any a-priori assumption concerning the return distribution or the PT-parameters.

- As a robustness check we have assessed our computational results also for the subset of investors for which the PT-parameters comply with the theoretical assumptions concerning these parameters and have found that all of our observations remain valid.

The rest of the paper is structured as follows. In Section 2 we formulate the CPT portfolio optimization problem, discuss it from the numerical point of view and present our proposal for an algorithm for solving this type of problems numerically. Section 3 presents the three proximity indices employed for the comparison of optimal portfolios obtained by the alternative approaches. The next section is devoted to describing and discussing the data-sets utilized in our numerical experiments. In the subsequent sections 5, 6 and 7 we present our numerical result obtained for the original data-set, for the data-set with an added European call option and for the data-set corresponding the associated normal distribution, respectively. Section 8 summarizes the numerical results in a comparative manner, concerning the three types of data-sets. Section 9 concludes.

2 The behavioral portfolio optimization models and numerical approaches for their solution

2.1 Problem formulations

The basic portfolio optimization model, based on (C)PT objective function maximization, can be formulated as:

$$\left. \begin{array}{ll} \max_{\lambda} & W(\lambda) := V(\xi^T \lambda) \\ & \mathbb{1}^T \lambda = 1 \\ & \lambda \geq 0, \end{array} \right\} \quad (1)$$

where ξ is the vector of asset returns, λ_i is the weight of the i^{th} asset in the portfolio, $i = 1, \dots, n$, $\mathbb{1}^T = (1, \dots, 1)$. V is an objective function corresponding to the (original) prospect theory (PT) or to the cumulative prospect theory (CPT). Due the nonnegativity constraint on the asset weights, short-sales are excluded. The optimal solution of the above problem will be denoted by λ^* .

Throughout this paper we consider the finitely distributed case of asset returns, given by a table of scenarios:

$$\begin{pmatrix} \xi^1, & \dots, & \xi^S \\ p_S, & \dots, & p_S \end{pmatrix}; \quad p_s > 0, \forall s \quad \text{and} \quad \sum_{s=1}^S p_s = 1.$$

For formulating the objective function of (1), we introduce a value function first, which plays a similar role as a utility function in expected utility theory. In this paper we will employ the piecewise-power value function of Kahneman and Tversky (1979) [19], which can be formulated as

$$v(x) = \begin{cases} (x - RP)^{\alpha^+}, & \text{if } x \geq RP \\ -\beta(RP - x)^{\alpha^-}, & \text{if } x < RP \end{cases}, \quad (2)$$

where RP is the reference point, α^+ and α^- are the risk-aversion parameters and β denotes the loss-aversion parameter. For these parameters $0 < \alpha^+, \alpha^- \leq 1$ and $\beta > 1$ are assumed to hold. Kahneman and Tversky (1979) have found the median parameter values $\alpha^+ = \alpha^- = 0.88$ and $\beta = 2.25$. According to the function (2), investors evaluate their gains and losses with respect to a reference point RP . In the gain domain they are risk-averse whereas in the loss domain the risk-seeking behavior prevails. Due to the requirement $\beta > 1$, in the loss domain the function is steeper as in the gain domain (provided that $\alpha^+ = \alpha^-$ holds).

In the original prospect theory PT, the objective function V in (1) is formulated in analogy with expected utility as

$$W_{PT}(\lambda) := V_{PT}(\xi^T \lambda) = \sum_{s=1}^S w(p_i) v((\xi^s)^T \lambda) \quad (3)$$

where, however, the probabilities are replaced by their distorted values, $p_i \mapsto w(p_i)$. The probability distortion function w overweighs small- and underweighs large probabilities thus modeling the observed investor behavior in this respect. We will use the original probability weighting function of Kahneman and Tversky which can be formulated as

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}} \quad (4)$$

with $0 < \gamma \leq 1$. According to the experiments of Kahneman and Tversky, the median value for γ is $\gamma = 0.65$. However, as observed by Rieger and Wang (2006) [30] and Ingersoll (2008) [17], this function loses the desired

property of strict monotonic increase for parameter values $\gamma < 0.278$.

For formulating the objective function for the cumulative prospect theory for a fixed λ , the vector of portfolio returns $((\xi^1)^T \lambda, \dots, (\xi^S)^T \lambda)$ has to be sorted first in increasing order. Let $((\eta^1)^T \lambda, \dots, (\eta^S)^T \lambda)$ denote the sorted vector, for which

$$(\eta^1)^T \lambda \leq \dots \leq (\eta^t)^T \lambda \leq RP \leq (\eta^{t+1})^T \lambda \leq \dots \leq (\eta^S)^T \lambda$$

holds and let $(\bar{p}_1, \dots, \bar{p}_S)$ be the correspondingly sorted vector of probabilities. Then the objective function is computed according to

$$W_{CPT}(\lambda) := V_{CPT}(\xi^T \lambda) = \sum_{i=1}^S \pi_i v((\eta^i)^T \lambda) \quad (5)$$

where the weights π_i are determined on the basis of probability weighting functions w^+ and w^- as

$$\begin{aligned} \pi_1 &= w^-(\bar{p}_1) \quad \text{and} \quad \pi_i = w^-(\bar{p}_1 + \dots + \bar{p}_i) - w^-(\bar{p}_1 + \dots + \bar{p}_{i-1}) \quad \text{for } 2 \leq i \leq t; \\ \pi_S &= w^+(\bar{p}_S) \quad \text{and} \quad \pi_j = w^+(\bar{p}_j + \dots + \bar{p}_S) - w^+(\bar{p}_{j+1} + \dots + \bar{p}_S) \quad \text{for } t < j \leq S-1, \end{aligned}$$

where the probability weighting functions are defined as

$$w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}} \quad \text{and} \quad w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}} \quad (6)$$

with $0 < \delta \leq 1$, $0 < \gamma \leq 1$. According to this construction, both tails of the probability distribution are distorted. Thus, while prospect theory overweighs small probabilities, cumulative prospect theory puts more weight on extreme events. Notice that without probability weighting ($\delta = \gamma = 1$) CPT reduces to PT, the latter also without probability weighting.

For detailed presentations of the various aspects of prospect theory see, e.g., Hens and Bachmann (2008) [15] or Wakker (2010) [34].

2.2 Numerical solution approaches

As mentioned before, Levy and Levy (2004) [23] suggest the following algorithm for solving the CPT optimization problem (1): generate a fine-enough mesh along the MV efficient frontier first. Subsequently compute the CPT objective function value for each of the corresponding portfolios and finally select a portfolio with the highest objective value. In our computational

experiments we took an equidistant subdivision concerning the expected value, consisting of 1000 subdivision intervals and computed the corresponding frontier-portfolios by solving the related MV optimization problems.

Concerning optimization algorithms for solving the portfolio selection problem (1) directly, this optimization problem turns out to be quite difficult from the numerical optimization point of view.

The prospect theory objective function W_{PT} is in general a non-smooth function, neither concave nor convex. For this case De Giorgi, Hens and Mayer (2007) [7] have developed an algorithm based on spline-smoothing of the value function v for achieving differentiability and subsequent application of a general purpose nonlinear programming solver. The non-convexity of the optimization problem has been dealt with by applying a multi-start method, based on randomly generated points over the unit simplex and taking the best of these points as a starting point for the solver, in an iterative manner.

The same approach does not work in the CPT case. Even if the value function v would be smooth, the CPT objective function V_{CPT} remains non-smooth, since the computation of it involves a sorting of the portfolio-return realizations, where the ordering will depend on λ .

For illustrating the difference between original and cumulative prospect theory we take an artificial example with 2 assets and 3 scenarios:

probability	Asset1	Asset2
0.5	-0.035	-0.01
0.2	0.03	-0.07
0.3	-0.025	0.01

The corresponding PT and CPT objective functions are displayed in Figure 1. We observe that the sorting of scenarios, inherent in the computation of the CPT value function, introduces additional non-smoothness.

Observe, however, that the feasible domain of our optimization problem (1) is the unit simplex. For economic equilibrium problems involving the unit simplex, a widely used method for computing fixed points is based on simplicial partitions. Our idea was to take a method of this type and apply it for the solution of the portfolio optimization problem (1). We have chosen the method of Eaves (1971) [10] and have implemented it in an adaptive simplicial refinement procedure, in a multi-start framework. The main steps in the algorithm are as follows:

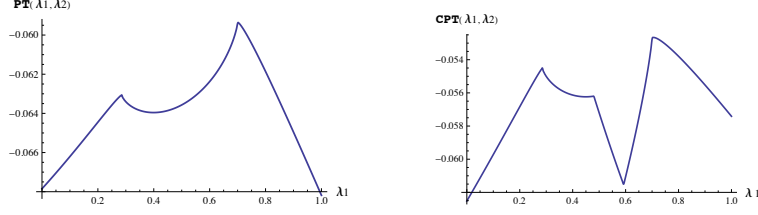


Figure 1: Example: The objective functions corresponding to the PT and to the CPT approach, respectively; $RP = 0$, $\alpha^+ = \alpha^- = 0.88$, $\beta = 2.25$, $\gamma = 0.65$

- *S1*: A uniform simplex grid is computed for the current simplex. With a grid constant (also called grid resolution) gc , this results in an equidistant subdivision of the edges into gc intervals.
- *S2*: A grid point λ^* with the highest objective function value is determined across the grid.
- *S3*: Next we consider a coarser subdivision (with grid resolution $\overline{gc} \sim 0.5 \cdot gc$, e.g.) and a sub-simplex of this subdivision which contains λ^* is determined by employing the method of Eaves.
- *S4*: This sub-simplex becomes the current simplex and the procedure is repeated by starting with *S1* anew.

We have implemented a combination of the basic algorithm with simulation. The idea is that instead of the unit simplex, the basic procedure is carried out on a sub-simplex found by simulation. This runs as follows.

We generate N uniformly distributed points in the unit simplex and on its boundary, and determine the best point $\tilde{\lambda}$ over this set of points. Then we compute the sub-simplex with grid resolution \overline{gc} , which contains $\tilde{\lambda}$. Subsequently the basic procedure is carried out with this sub-simplex as the starting current simplex. Thus the basic procedure is embedded into an outer simulation loop, with a prescribed number of the outer simulation steps. The outer loop is needed because our objective function to be maximized is non-concave; the resulting overall algorithm belongs to the class of multi-start random search methods, see, e.g., Törn and Zilinskas [32].

In our computational experiments we have chosen $gc = 8$ for the grid resolution and applied 4 adaptive sets. The outer simulation loop involved 20 starts, with a sample-size of 1000 for computing the starting point for each individual start.

Concerning regular grids over simplices respectively hypercubes, the number of equidistant subintervals of the edges of the simplex respectively the hypercube is called the grid constant and will be denoted by gc . The number of grid-points of a simplicial grid with grid constant gc and dimension n is $\binom{gc+n-1}{gc}$ whereas the number of grid-points of a rectangular grid with the same grid-constant is $(gc+1)^n$. The latter grows much more rapidly with increasing dimension; as an illustration see Table 1. Thus, the adaptive grid method is better suited for the multiple assets case but it clearly has its limitation concerning the number of assets, with a practical upper bound of around 10–15 assets.

n :	1	2	3	4	5	6	7	8	9	10
simplex:	9	45	165	495	1287	3003	6435	12870	24310	43758
cube:	9	81	729	6561	59049	531441	4782969	43046721	$3.87 \cdot 10^8$	$3.49 \cdot 10^9$

Table 1: The number of grid points for simplicial and rectangular grids, corresponding to grid-constant $gc = 8$ and varying dimension n

3 Comparing optimal portfolios obtained via the different approaches/objective functions

The majority of papers comparing expected utility maximization with the mean-variance (MV) method employ the following approach: the expected utility is maximized along the MV-efficient frontier and the portfolio obtained this way is used in the comparison. We apply this technique in the prospect theory optimization context. In our case this means that the following problem is solved:

$$\left. \begin{array}{l} \max_{\lambda} \quad V(\xi^T \lambda) \\ \lambda \in \Lambda_{MV}^* \end{array} \right\} \quad (7)$$

with Λ_{MV}^* denoting the set of portfolios along the MV efficient frontier. Let λ_{MV}^* stand for an optimal solution of the above problem. λ^* being an optimal solution of our prospect theory optimization problem (1), the inequality $V(\xi^T \lambda_{MV}^*) \leq V(\xi^T \lambda^*)$ clearly holds, since we have

$$\Lambda_{MV}^* \subset \{\lambda \mid \mathbb{1}^T \lambda = 1, \lambda \geq 0\}.$$

In practice, the optimization along the MV efficient frontier is carried out by employing a fine enough mesh over the frontier.

For details concerning the method see, e.g., Kroll, Levy and Markowitz (1984) [21], Levy and Levy (2004) [23], De Giorgi and Hens (2009) [6].

Having computed portfolios via solving the prospect theory portfolio optimization problem (1) for PT and CPT and having also obtained an optimal portfolio via solving (7), the question arises how to compare them? We need some kind of a “similarity” or “proximity” measure for that.

Recall that λ^* denotes the optimal portfolio obtained via maximizing a (C)PT objective function according to (1) and λ_{MV}^* stands for the MV–portfolio in the comparison, obtained by solving (7).

In the expected utility framework, Kroll, Levy and Markowitz (1984) [21] employ the objective functions ratio

$$I_{OBJR} := \frac{\mathbb{E}[u(\xi^T \lambda_{MV}^*)] - \mathbb{E}[u(\xi^T \lambda_{naive})]}{\mathbb{E}[u(\xi^T \lambda^*)] - \mathbb{E}[u(\xi^T \lambda_{naive})]},$$

where λ_{naive} is the “naïve” portfolio with equal weights. This is introduced for avoiding the dependence of the objective ratio on possible shifts in u (u and $u + C$ are equivalent concerning the ordering on the basis of expected utility, for any $C \in \mathbb{R}$). Observe that for the denominator $\mathbb{E}[u(\xi^T \lambda^*)] - \mathbb{E}[u(\xi^T \lambda_{naive})] \geq 0$ holds since λ^* is an optimal solution of the maximization problem (1) and λ_{naive} is a feasible solution of that problem. In applications of the above index it is assumed that the denominator is strictly positive. It is also presupposed that the numerator is nonnegative, that is, it is assumed that the optimal portfolio along the MV–frontier has at least as high objective value as the naïve portfolio. Under these circumstances the inequality

$$0 \leq I_{OBJR} \leq 1$$

holds, with larger values reflecting higher proximity.

Let us note that the comparative study of DeMiguel, Garlappi and Uppal (2009) [8] draws a new light on and emphasizes the actuality and importance of the I_{OBJR} proximity index as defined above.

Interestingly, Rios and Sahinidis (2010) [31] employ a ratio of the above type to assess differences between portfolios obtained by a globally convergent al-

gorithm versus portfolios obtained via local search.

We will employ this index as one of our comparison indices, adapted to our case as

$$I_{OBJR} = \frac{V(\xi^T \lambda_{MV}^*) - V(\xi^T \lambda_{naive})}{V(\xi^T \lambda^*) - V(\xi^T \lambda_{naive})},$$

where $V = V_{PT}$ and $V = V_{CPT}$ holds in the PT–MV and CPT–MV comparisons, respectively. We assume strict positivity of the denominator. In the PT–context we do not require nonnegativity of the numerator; negative index values just reflect a large dissimilarity of the portfolios. For the PT versus CPT comparison we take

$$I_{OBJR} = \frac{V_{PT}(\xi^T \lambda_{CPT}^*) - V_{PT}(\xi^T \lambda_{naive})}{V_{PT}(\xi^T \lambda_{PT}^*) - V_{PT}(\xi^T \lambda_{naive})},$$

where λ_{PT}^* and λ_{CPT}^* are the optimal PT and CPT portfolios, respectively. In reporting our computational results, values of I_{OBJR} will be presented in terms of percentages.

In the expected utility context a natural idea is to use certainty equivalents for the comparison since they are invariant under affine–linear transformations of u and have a direct economic interpretation (they are also called “cash–equivalents”). In analogy with the expected utility approach, in our prospect theory context a natural way to define the certainty equivalent CE_v of a portfolio λ is according to the equation:

$$v(CE_v[\xi^T \lambda]) = V(\xi^T \lambda) \Leftrightarrow CE_v[\xi^T \lambda] = v^{-1}(V(\xi^T \lambda)).$$

Recall from prospect theory that v must be strictly monotonically increasing thus the inverse involved in the above equation exists. This notion of a certainty equivalent has been employed for comparative purposes e.g. by Døskeland and Nordahl (2006) [9] and by De Giorgi and Hens (2009) [6].

For expected utility Pulley (1983) [26] suggests utilizing the ratio

$$\frac{CE_u[\xi^T \lambda_{MV}]}{CE_u[\xi^T \lambda_u]} = \frac{u^{-1}(\mathbb{E}[u(\xi^T \lambda_{MV})])}{u^{-1}(\mathbb{E}[u(\xi^T \lambda_u)])}$$

for the comparison. Reid and Tew (1986) [28] observe empirically that concerning their data–set there is not much difference between I_{OBJR} and the above index.

Kallberg and Ziemba (1983) [20] compare empirically the optimal portfolios obtained via expected utility maximization involving different utility functions, the analysis not being focused on the comparison with MV. Their proximity index is constructed as follows. Let λ^1 and λ^2 be optimal portfolios obtained via expected utility maximization, according to two different utility functions. One of the two utility functions is selected as reference utility, let us denote this utility function by u . Kallberg and Ziemba employ as proximity measure a normalized difference based on certainty equivalents:

$$I_{CER} := \frac{CE_u[\xi^T \lambda^1] - CE_u[\xi^T \lambda^2]}{CE_u[\xi^T \lambda^1]} = \frac{u^{-1}(\mathbb{E}[u(\xi^T \lambda^1)]) - u^{-1}(\mathbb{E}[u(\xi^T \lambda^2)])}{u^{-1}(\mathbb{E}[u(\xi^T \lambda^1)])}.$$

In the prospect theory case, with certainty equivalents being expressed in terms of net returns, they may have negative values and the denominator in the above expression may be close to zero or it may even be zero. Therefore, in this paper we work with certainty equivalents in terms of gross returns. We apply the above formula based on the prospect theory value function v :

$$I_{CER} = \frac{CE_v[\xi^T \lambda^1] - CE_v[\xi^T \lambda^2]}{CE_v[\xi^T \lambda^1]}$$

where we assume that in gross return terms the denominator is positive. In the above expression λ^1 is an optimal portfolio obtained by solving (1) in a PT or CPT setting whereas λ^2 is either the optimal portfolio regarding the other prospect theory variant or regarding MV. Consequently, we have the inequality

$$0 \leq I_{CER},$$

with higher values corresponding higher dissimilarity. Notice the difference with respect to the index I_{OBJR} , for which higher values indicated higher similarity.

Similarly as for I_{OBJR} , we report our computational results related to I_{CER} in terms of percentages.

DeMiguel, Garlappi and Uppal (2009) [8] and also De Giorgi and Hens (2009) [6] utilize the difference of the certainty equivalents for assessing difference in the portfolio allocations according to PT and MV:

$$I_{CED} := CE_v[\xi^T \lambda^*] - CE_v[\xi^T \lambda_{MV}^*]$$

and interpret the difference as added value in monetary terms. We employ this index for comparing two portfolios in the form:

$$I_{CED} = CE_v[\xi^T \lambda^1] - CE_v[\xi^T \lambda^2],$$

where λ^1, λ^2 have the same interpretation as for the index I_{CER} and we have also here

$$0 \leq I_{CED},$$

with higher values indicating higher dissimilarity.

In reporting our computational results, I_{CED} -values will be reported as annualized returns, in percentage terms (recall that in the original data-set monthly returns are given).

The paper of DeMiguel, Garlappi and Uppal [8] is an example for using statistical techniques in comparing portfolios: an out-of-sample analysis with a moving window. In general, out-of-sample analysis is a further possibility for comparing the performance of different portfolios; this approach will not be employed in the present paper. Let us remark that in the above mentioned paper, besides certainty equivalents, Sharpe-ratios are also utilized for comparison purposes.

4 The data-sets

The BhFS benchmark data-set consists of monthly net returns for 8 asset-classes (indices) for the time period 02.1994 – 05.2011, thus resulting in a sample-size of 208 elements. The indices included are listed in Table 2; a summary statistics concerning the data-set is shown in Table 12 along with the empirical correlation matrix in Table 13 in the Appendix.

In the sequel we will repeatedly argue that the main cause of observed phenomena is the fact that the normality assumption does not hold in our case. For testing the null-hypothesis that our data-set can be considered as a sample from a multivariate normal distribution, we have carried out Mardia's skewness and kurtosis tests for multivariate normality (for these tests see, e.g., Rencher (2002) [29]). Both for multivariate skewness and for multivariate kurtosis we have got the p -values $p = 0.000$, see Table 14 in the Appendix. Thus the null-hypothesis can clearly be rejected on the basis of both tests, on a 99.9% significance level.

In our numerical tests we will work with three data-sets.

GSCITR	Goldman Sachs Commodity Index; total return;
I3M	3 months US Dollar LIBOR interest rate;
HFRIFFM	Hedge Fund Research International, Fund of Funds, market defensive index;
MSEM	Morgan Stanley Emerging Markets Index, total return, stocks;
MXWO	MSCI World Index, total return, stocks (developed countries);
NAREIT	FTSE, US Real Estate Index, total return;
PE	LPX50, LPX Group Zurich, Listed Private Equities, total return;
JPMBD	JP Morgan Bond Index, Developed Markets , total return.

Table 2: The indices included in the monthly-returns data-set

4.1 First (original) data set

When working with samples, all scenarios are equally likely and in (C)PT-based portfolio optimization the optimal portfolios will be the same for all probability-weighting parameter values in PT, whereas they will depend on those parameter values in CPT. Therefore, since we also want to explore the influence of probability weighting in the comparison of PT and CPT, we have computed an empirical distribution (a lottery) having 15 realizations, on the basis of the sample. Keeping in mind that we wished to avoid distributional assumption as concerning our data, for this purpose we have utilized k -means clustering with the Manhattan-distance. Concerning the k -means clustering approach see, e.g., Everitt et al. (2011) [11] and the references therein. The resulting empirical distribution is displayed in Table 15 in the Appendix, page 45, this is our **first data-set** which we will also call as original data-set.

4.2 Second data set, involving a call option

The particular choice of $k = 15$ was motivated by our aim to get a **second data-set** by appending the original data-set by a European call option. The reason for taking also a data-set of this type is twofold: On the one hand, in testing the method of Levy and Levy we wished to take also a data-set with a clear deviation from the normality assumption. On the other hand, we aim also to test the positive skewness-loving behavior of CPT investors, as discussed in Barberis and Huang (2008) [2]. We have appended the original data-set a European call option on the index **MXWO**. To achieve this, we have proceeded as follows.

With a given data-set, consisting of scenarios and corresponding probabilities, we consider this as an incomplete market which should be arbitrage-free

for being able to append it with a call option. For testing this we compute state prices first, by solving the following linear programming problem

$$\left. \begin{aligned} \max_{\pi, \varepsilon} \quad & \varepsilon \\ & \sum_{s=1}^S \pi_s = 1 \\ & \sum_{s=1}^S r_s^k \pi_s = r_f, \quad k = 1, \dots, K \\ & \pi_s \geq \varepsilon, \quad s = 1, \dots, S \end{aligned} \right\} \quad (8)$$

where r_f is the risk free rate. Let (π^*, ε^*) be an optimal solution. The above problem serves for computing state prices with the smallest state price being maximal. The market is arbitrage-free if for the optimal solution $\varepsilon^* > 0$ holds. Concerning state prices and incomplete markets see, e.g., Hens and Rieger (2010) [16] or Černý (2009) [5].

In our case we have $K = 8$ assets and wished to find an arbitrage-free data-set for the monthly risk-free rate $r_f = 0.002$. For achieving this we have performed computational experiments by generating empirical distributions with an increasing number of scenarios S by k -means clustering ($k = S$) as applied to our sample consisting of 208 elements. For each of these empirical distributions we have solved (8) and finally for $S = 15$ we have obtained positive state prices. This empirical distribution with $S = 15$ realizations and $K = 8$ assets (see Table 15 in the Appendix) served as the basic data-set of our empirical comparisons in Section 5, cf. also Section 4.

Using the state prices obtained via solving (8), we add a call option to our data-set, on the 6th index MXWO. The data matrix is appended by a column corresponding to the call option, with the entries scaled such that for the added column \hat{r} the relation $\sum_{s=1}^S \hat{r}_s \pi_s^* = r_f$ holds thus guaranteeing the arbitrage-free nature of the new data-set. As a strike price (in terms of net returns) we choose 0.1 which ensures that there is a positive payoff in only one state $s = 12$ (see Table 15) this way producing a clear deviation from the normality assumption. Thus in the added column \hat{r} the only nonzero element is $\hat{r}_{12} = 0.283$ with the specific value computed on the basis as discussed above. Consequently, apart of the distributional assumptions, our setting is quite similar to the setting in Barberis and Huang (2008) [2].

4.3 Third data set, corresponding to a normal distribution

For comparative reasons we have also generated a **third data-set** for testing the effects of a normality assumption. This data-set has been generated as follows. Taking the empirical expected value vector μ and empirical covariance matrix Σ of the original scenarios, we simulated a sample consisting of 10'000 elements from the corresponding multivariate normal distribution, by employing the standard method based on the Cholesky-factorization of Σ . Subsequently we employed k -means clustering to get 15 scenarios with corresponding probabilities.

Just for cross checking the quality of our generated sample and Mardia's multivariate normality test, we have carried out Mardia's skewness and kurtosis tests (see, e.g., Rencher [29]) for the sample for testing the null-hypothesis that the generated sample can indeed be considered as a sample from a multivariate normal distribution. For the sample with 10'000 elements we obtained the following results: For the skewness-test we have got the p -value $p_{Skew} = 0.85$ and for the kurtosis test $p_{Kurt} = 0.31$. We have also carried out the tests with the first 208 sample elements and have obtained $p_{Skew} = 0.80$ and $p_{Kurt} = 0.17$, see Table 14 in the Appendix. Thus, for both sample sizes and both on the basis of the skewness and kurtosis tests, we clearly fail to reject the null-hypothesis, even on a 90% level of significance.

4.4 Data set for the PT parameter settings

Concerning the parameters of the PT value function and of the probability weighting functions, we utilize the settings published by Abdellaoui, Bleichrodt and Paraschiv (2007) [1]. In Appendix C of this paper the authors present the results of an experimental elicitation of the PT-parameters for 48 subjects. The reference point is 0 throughout. Concerning loss aversion, several parameter settings are presented according to the different definitions of loss-aversion. For our numerical experiments we have chosen the classical Kahneman-Tversky loss-aversion coefficient, corresponding to the median estimator.

Regarding probability distortion, the authors present probabilities p_g and p_l for gains and losses, respectively, having the properties $w^+(p_g) = 0.5$ and $w^-(p_l) = 0.5$. For determining the parameters δ and γ in the probability distortion functions (6), the corresponding equations have to be solved for the

parameters δ and γ , for fixed probability p . These equations have a unique solution for $p \geq 0.5$ while for $p < 0.5$ we have computed the solutions with the least absolute deviations in the equations. Since obviously there is no need for a high-precision solution, we employed straightforward grid search for determining δ and γ . For the case of ordinary prospect theory, we have employed the parameter γ of w^+ for the probability distortion.

All of our computations are carried out in turn for all of the 48 settings, with the subjects participating in the experiment considered as separate investors in this respect.

5 The original data-set: Comparing the optimal portfolios

In this section we report our numerical results obtained on the basis of the empirical distribution corresponding to the original data-set. We have solved the portfolio optimization problem (1) both for the PT and for the CPT objective function. Subsequently we have also maximized both objective functions along the mean-variance frontier by taking an equidistant subdivision concerning the expected value and consisting of 1000 subdivision intervals. All computations have been carried out in turn for all of the subjects (considered as investors) by utilizing the PT value-function parameters as listed in the paper [1].

For the PT objective function in (1), the optimal portfolios can be seen in Figure 32 whereas those for the CPT objective function are displayed in Figure 33.

Concerning the optimization along the MV-frontier, the optimal portfolios for the PT objective function are displayed in Figure 34. For the CPT objective function the optimal portfolios can be seen in Figure 35.

Next we compare the portfolios obtained, by employing the proximity measures I_{OBJR} , I_{CED} and I_{CER} , defined and discussed in Section 3. The computational results are listed according to the relations PT versus CPT, PT versus MV and CPT versus MV. For each of these relations a summary statistics along with frequency-histograms is presented. In addition, for the relation concerning MV, in the $\mu - \sigma$ space the optimal portfolios are displayed along with the efficient frontier, as well as a histogram concerning

the distances to the MV–frontier, see Table 3, Figure 2, Table 4, Figure 3, Figure 5, Table 5, Figure 7, Figure 9.

As discussed in Section 3, values of all of the three proximity indices will be reported in percentage terms, with I_{OBJR} and I_{CER} being dimensionless ratios and I_{CED} having the dimension of annual returns.

	I_{OBJR}	I_{CED}	I_{CER}
Min.	-92.65	-0.356	-0.030
Mean	89.50	0.562	0.046
Median	99.25	0.089	0.007
Max.	100.9	11.13	0.921
Std. Dev.	32.23	1.695	0.140
Skewness	-4.349	5.249	5.254
Kurtosis	22.99	32.24	32.30

Table 3: Summary statistics for the proximity indices, PT versus CPT

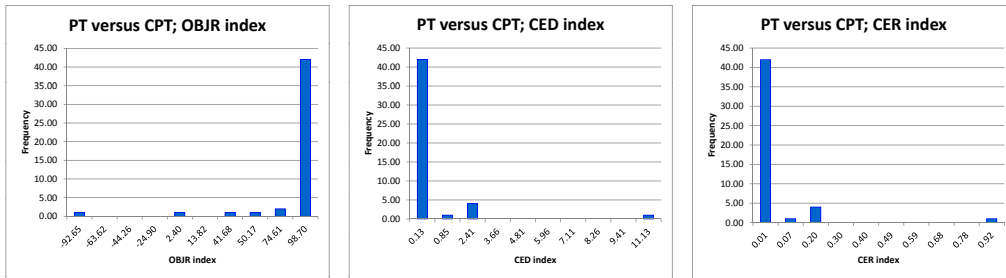


Figure 2: Histograms for the PT–CPT comparison

On the basis of the information in Table 3 and Figure 2 we observe that there is a substantial difference between the PT and CPT approaches. This conclusion is supported by the following fact: if we drop observations with outliers corresponding to very small and very large I_{CED} values, there still remain 25 investors (more than 50% of all investors) with I_{CED} –values between 0.01% and 0.45%, with an average of 0.17%, in terms of annualized returns. Neglecting those 11 investors for which at least one of the PT–assumptions $\alpha^+ \leq 1$, $\alpha^- \leq 1$ or $\beta > 1$ is violated, there still remain 22 investors with I_{CED} –values in the above range. On the first glance these numbers seem to be small, but in the wealth management field they count as quite large (between 1bp and 45 bps), see the discussion and example in De Giorgi and Hens (2009) [6]. This way our numerical experiments support the results of the comparative study of Fennema and Wakker (1997) [12].

Next we discuss the numerical results concerning the PT–MV relation, see Table 4 and Figure 3. As for the PT–CPT relation we observe a substantial deviation between the results from the two different approaches. Taking again a middle range of the I_{CED} values, for 25 investors now we have the range from 0.12% to 1.2% and average value of 0.6% which clearly indicates a substantial added value which can be obtained by investing into the PT–optimal portfolio instead of the portfolio which is optimal along the MV–frontier. The values obtained for the other two proximity indices I_{OBJR} and I_{CER} clearly support this conclusion. When dropping the 11 investors with not PT–conform parameter values, in the middle range we still have 22 investors with $0.07\% < I_{CED} < 0.98\%$ and average I_{CED} value 0.4%.

	I_{OBJR}	I_{CED}	I_{CER}
Min.	-13.80	0.002	0.0002
Mean	81.24	3.536	0.273
Median	98.02	0.429	0.036
Max.	99.88	83.85	6.206
Std. Dev.	34.49	13.30	0.994
Skewness	-1.806	5.158	5.057
Kurtosis	4.539	29.67	28.61

Table 4: Summary statistics for the proximity indices, PT versus MV

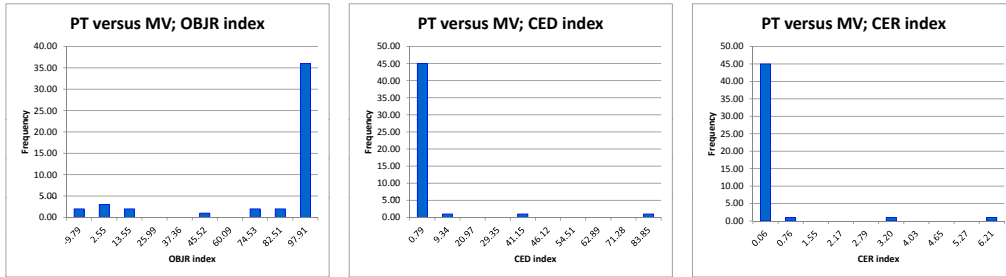


Figure 3: Histograms for the PT versus MV comparison

Concerning the difference between the results from the two approaches, the question arises what may be the cause of this phenomenon? As mentioned before, Barberis and Huang (2008) [2] proved under appropriate assumptions that CPT–investors prefer positively skewed securities. Figure 4 displays the skewness of the optimal portfolios obtained via the PT optimization approach and of those computed by maximizing the PT objective function along the MV frontier the latter denoted by PT(MV). It can be seen that the PT–optimal portfolios are less negatively skewed and some of them are even

positively skewed in comparison with the PT(MV) portfolios.

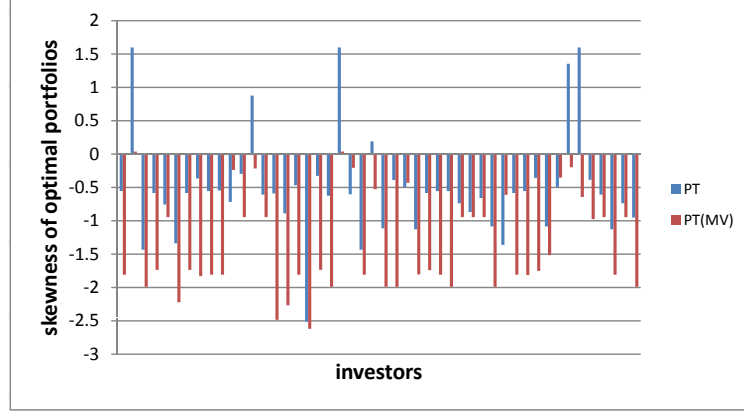


Figure 4: The skewness of the optimal portfolios obtained via the PT and the PT(MV) approach.

As an additional proximity measure between the portfolios obtained via the two approaches we consider the distance of the PT-optimal portfolios to the mean-variance frontier, see Figure 5. Since in the figure displaying the MV frontier the points corresponding to the (σ, μ) values of the optimal portfolios may represent several portfolios, a more detailed consideration is needed. To investigate this phenomenon we have checked the PT-parameter-dependence of the distance and have found that the points located on a larger distance correspond to low values of the loss-aversion parameter β , see Figure 6.

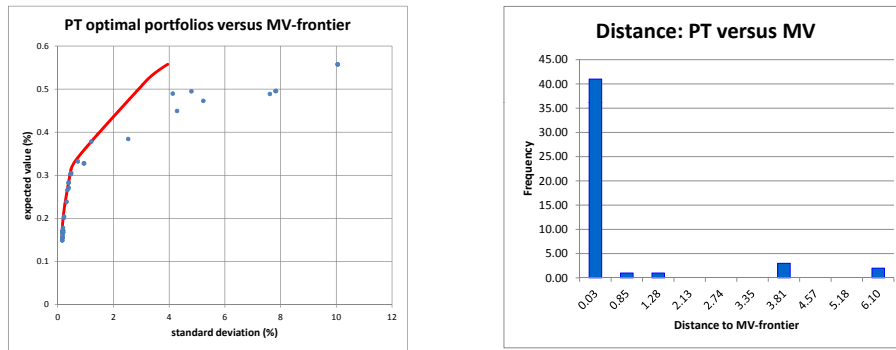


Figure 5: Optimal PT-portfolios versus MV frontier. On the right-hand-side the frequency-histogram of the distances to the MV-frontier is displayed.

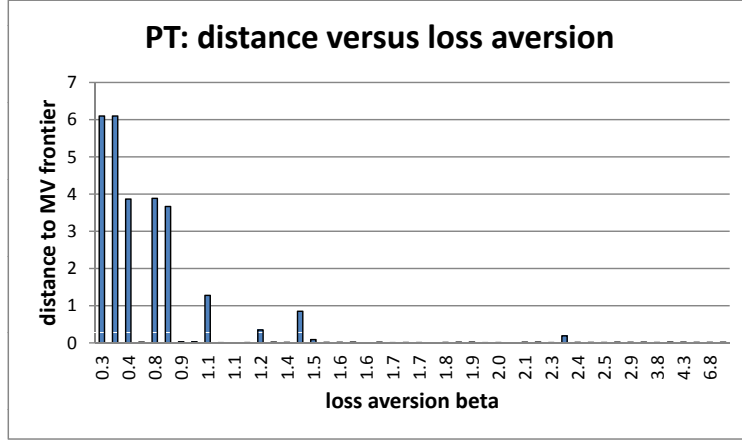


Figure 6: PT: the dependence of the distance to the MV-frontier on the loss-aversion parameter β .

When neglecting the 8 portfolios corresponding to investors with loss-aversion parameter $\beta < 1$, we get the average distance value 0.08% which is still definitely relatively large.

Finally we consider the CPT versus MV relation, for the numerical results see Table 5 and Figure 7. Similarly as for the previous two cases, we see that there is a considerable deviation between the results from CPT-optimization and from the CPT(MV) approach. Regarding the relation between the CPT-optimal portfolios and those obtained by maximizing the CPT objective function along the MV-frontier, we have in terms of annual returns on the average a remarkable $I_{CED} = 2.2\%$ deviation regarding certainty equivalents, with a maximum value of $I_{CED} = 67.9\%$, minimal value of $I_{CED} = 0.003\%$, standard deviation 9.9% and positive skewness.

	I_{OBJR}	I_{CED}	I_{CER}
Min.	-18.46	0.0028	0.0002
Mean	76.89	2.1838	0.171
Median	97.07	0.1734	0.014
Max.	99.43	67.87	5.151
Std. Dev.	38.15	9.877	0.751
Skewness	-1.544	6.265	6.207
Kurtosis	3.604	41.61	41.06

Table 5: Summary statistics for the proximity indices, CPT versus MV

Choosing again a middle range of the I_{CED} values, for 25 investors now we have the range from 0.05% to 0.78% and average value of 0.26% which

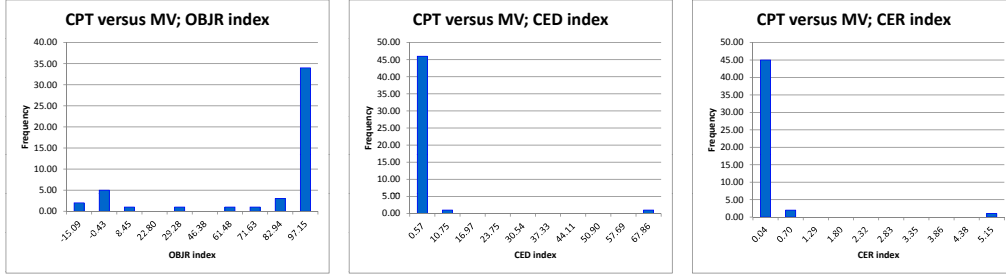


Figure 7: Histograms for the CPT–MV comparison

is a clear indication of the difference concerning the results from the two approaches. As for the previous two relations PT versus CPT and PT versus MV, the numerical results obtained for the other two proximity indices fully support this conclusion. Dropping the 11 investors with not PT theory conform parameter values, we have now in the middle range $0.03\% < I_{CED} < 0.35\%$ and an average of 0.14% thus indicating a difference between the two approaches also in this case.

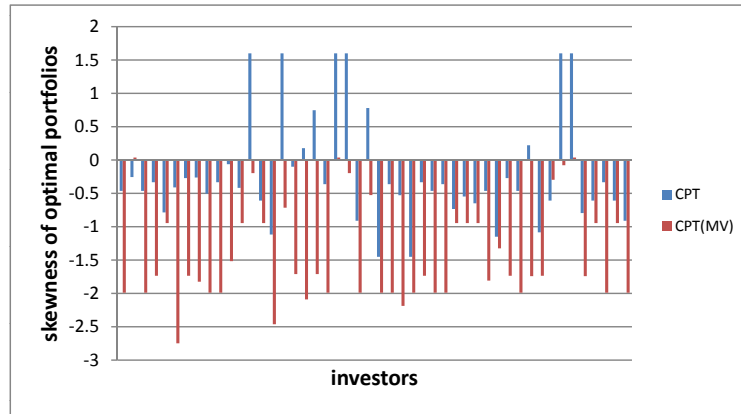


Figure 8: The skewness of the optimal portfolios obtained via the CPT and the CPT(MV) approach.

Regarding the positive skewness preference of CPT–investors, Figure 8 displays the skewness of the optimal portfolios obtained via the CPT optimization approach and of those computed by the CPT(MV) method. Similarly as in the PT case, the skewness of the CPT–optimal portfolios is larger than the skewness of the portfolios in the MV–case with some of them having a positive skewness.

The distances between the (σ, μ) values of the optimal CPT-portfolios and the MV-frontier indicate also in this case that the former do not lie along the MV frontier, see Figure 9.

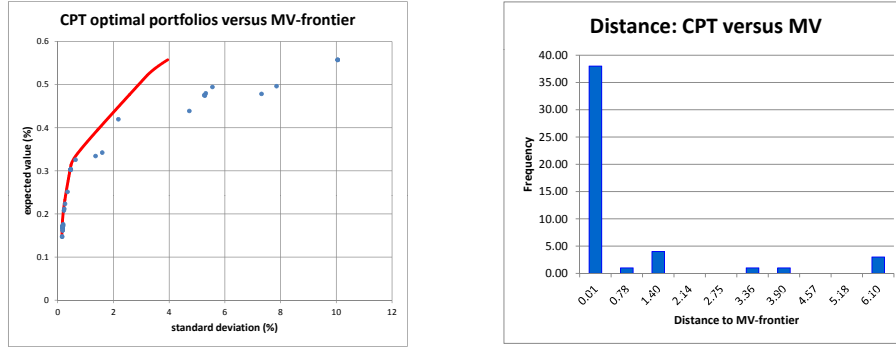


Figure 9: Optimal CPT-portfolios versus MV frontier. On the right-hand-side the frequency-histogram of the distances to the MV-frontier is displayed.

For checking the possible causes, we investigated again the PT-parameter-dependence of the distance and found that the largest distances correspond to low values of the loss-aversion parameter β , see Figure 10.

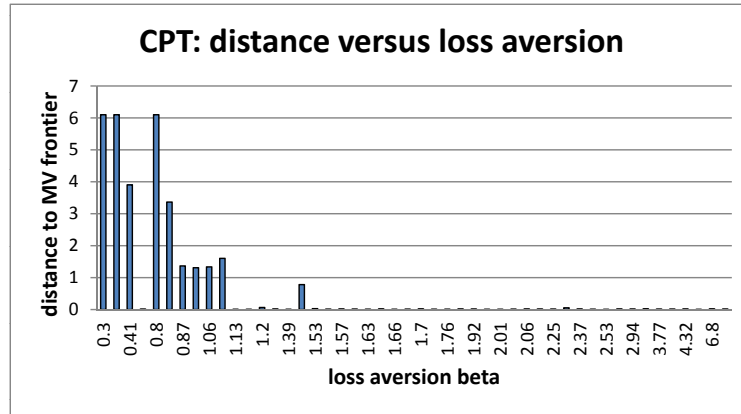


Figure 10: CPT: the dependence of the distance to the MV-frontier on the loss-aversion parameter β .

Dropping the 8 portfolios corresponding to loss-aversion parameter $\beta < 1$, we have the average distance value 0.53% which is definitely non-zero and for this subset of investors substantially larger than for the PT-case.

Summarizing: Based on our computational results we conclude that the PT and CPT portfolios differ substantially from each other and from the portfolios obtained by maximizing the PT respectively CPT objective functions along the MV-frontier. Thus our computational results support the findings of Fennema and Wakker (1997) [12] concerning the PT–CPT relation and harmonize with the findings of Barberis and Huang (2008) [2] concerning CPT–investor preferences for skewness. On the other hand, it turned out that the algorithm suggested by Levy and Levy (2004) [23] for computing CPT–optimal portfolios (extended by Pirvu and Schulze (2012) [25]) substantially depend on the distributional assumption under which the results were derived. In particular, the hypothesis that the CPT–optimal portfolios are located along the MV frontier does not hold in our case. Our numerical results also provide an extension to the empirical study of De Giorgi and Hens (2009) [6], by taking into account probability weighting in the original prospect theory case.

Interestingly, the difference between PT and CPT is noticeably smaller than the difference with respect to MV for both PT and CPT.

Notice that, due to the constraint for excluding short sales, the mean–variance frontier has points of non–smoothness, where the set of assets with positive weights changes. One such point is at $\mu = 0.31$, for higher μ values the (μ, σ) values of the (C)PT optimal portfolios deviate from the MV–frontier, the deviation being increased with increasing μ .

6 Employing a data–set with an added European call option

This section is devoted to report on our computational results for the data–set appended by a European call option on the index **MXWO**. In carrying out the computational experiment we proceed analogously as in the previous Section 5.

As in the previous section, values of the proximity indices are reported in percentage terms, with I_{CED} expressed in terms of annual returns.

On the basis of summary statistics the numerical results for the comparison of the PT and CPT approaches, based on the proximity indices, are quite similar to those obtained with the original data–set (c.f. Figure 3),

	I_{OBJR}	I_{CED}	I_{CER}
Min.	-56.98	0	0
Mean	88.32	0.501	0.041
Median	99.42	0.034	0.003
Max.	100	2.853	0.235
Std. Dev.	28.53	0.780	0.064
Skewness	-3.507	1.621	1.610
Kurtosis	16.12	4.598	4.551

Table 6: Call option added: Summary statistics for the proximity indices, PT versus CPT

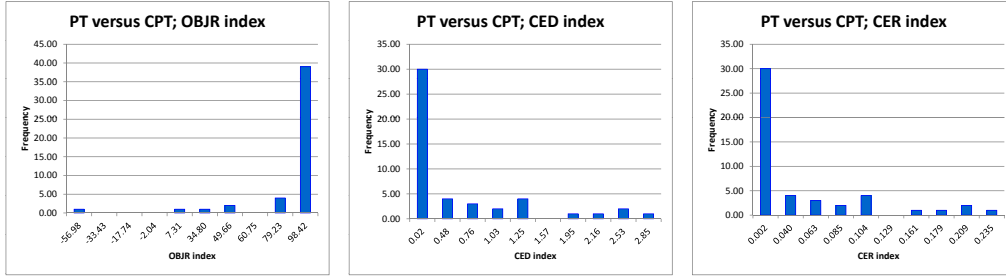


Figure 11: Call option added: Histograms for the PT–CPT comparison

see Figures 6 and 11 and indicate a sizeable difference. With this second data-set there is a noticeable difference, however: 21 out of the 48 investors invest their whole wealth into the call option, both in the PT and in the CPT case. The background of this behavior is skewness-loving and will be discussed later on. Nevertheless, for 17 investors (35% of all investors) we have $0.002\% < I_{CED} < 1.16\%$ with an average value of 0.7%. Consequently, apart of the joint property of skewness loving, the PT and CPT approaches prove to be different also on the basis of the second data-set. This conclusion is further justified by dropping the 11 investors with parameter values not satisfying the theoretical assumptions. When further neglecting 8 investors with the highest I_{CED} values, there remain 11 investors with $0.08\% < I_{CED} < 0.83\%$ and average value 0.48%.

The summary statistics for the PT–MV comparison is displayed in Table 7 and Figure 12; the results indicate again a sizeable difference between the two approaches. Comparing these with their counterparts for the first data-set (Table 4 and Figure 3), we observe a considerable increase in the difference between the PT and MV approaches. Taking again 25 investors in the middle range concerning I_{CED} values, we have $1.15\% < I_{CED} < 25.5\%$ with an average of 8.9% in annual returns and this makes up a huge difference regarding the two optimization approaches. This observation remains valid when we

	I_{OBJR}	I_{CED}	I_{CER}
Min.	-0.917	0.002	0.0002
Mean	75.74	13.75	1.096
Median	88.58	5.215	0.433
Max.	99.87	83.85	6.532
Std. Dev.	29.50	20.79	1.610
Skewness	-1.492	2.135	2.033
Kurtosis	3.681	7.024	6.559

Table 7: Call option added: Summary statistics for the proximity indices, PT versus MV

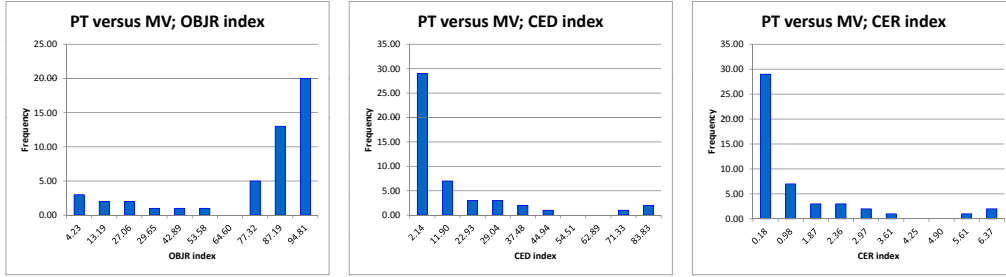


Figure 12: Call option added: Histograms for the PT–MV comparison

restrict our consideration to the 37 investors with PT–conform parameter values. In the middle range then we have $0.79\% < I_{CED} < 13.9\%$ with the average value 5.5%.

Figure 13 displays the skewness values of the optimal portfolios obtained by the two different approaches and clearly indicates that PT–investors exhibit a definitive preference for positive skewness. Indeed, 21 investors allocate their whole wealth into the call option and in addition 16 investors allocate a positive fraction of their wealth to the call option; see Figure 36 in the Appendix. The observation remains valid if we drop the 11 investors with not PT–conform parameter values. These numerical results indicate that the result of Barberis and Huang (2008) [2] concerning skewness–loving behavior of CPT–investors also holds for the PT–case. Notice that PT with probability weighting is obviously not a special case of CPT.

Next we discuss the numerical results concerning the distance of PT–optimal portfolios to the MV frontier, see Figure 14, which indicates that also in this case that most of the (σ, μ) values of the optimal PT–portfolios do not lie on the MV–frontier.

Analogously as for the original data set, the dependence of the distance from

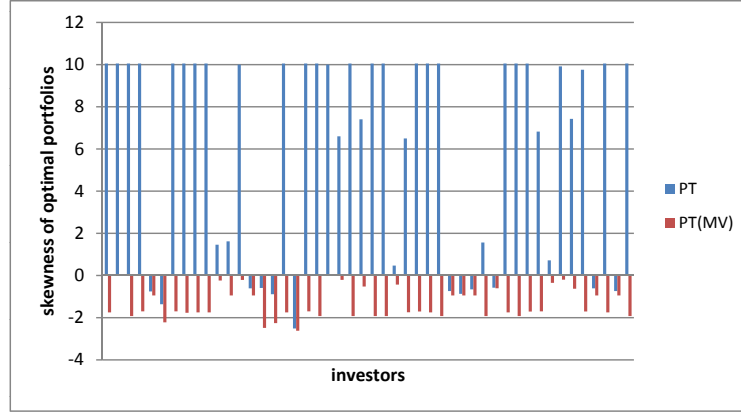


Figure 13: Call option added: The skewness of the optimal portfolios obtained via the PT and the PT(MV) approach.

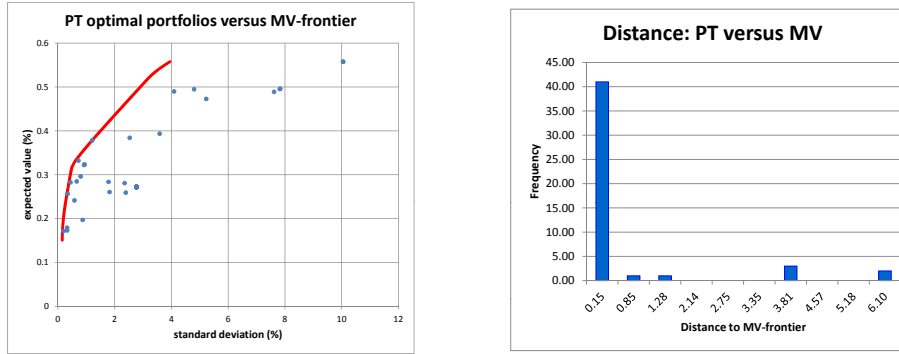


Figure 14: Call option added: Optimal PT-portfolios versus MV frontier. On the right-hand-side the frequency-histogram of the distances to the MV-frontier is displayed.

β is shown in Figure 15, indicating that the largest distances correspond to cases with loss-aversion $\beta < 1$.

Taking into account only the cases with $\beta \geq 1$, we obtain the average distance 0.2%. Comparing this with the average distance 0.08% shows that this distance has substantially increased. Thus, we may argue also in this case that the set of all PT-optimal portfolios is not a subset of the MV-frontier portfolios.

The computational results for the proximity indices, related to the CPT-MV comparison are shown in a summarized form in Table 8 and Figure 16.

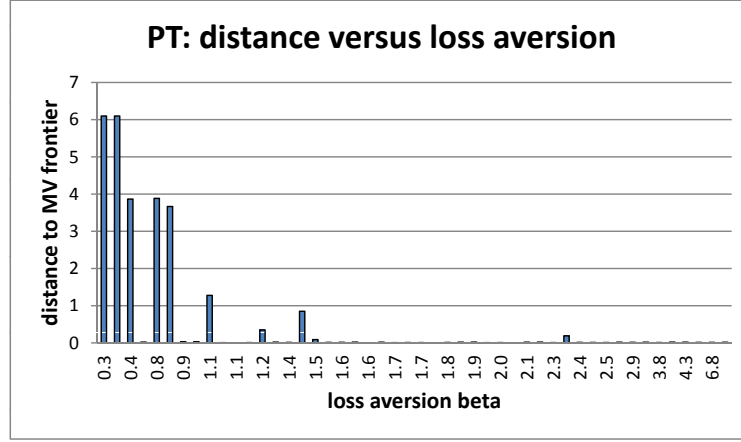


Figure 15: Call option added: the dependence of the distance to the MV–frontier on the loss–aversion parameter β , for PT.

	I_{OBJR}	I_{CED}	I_{CER}
Min.	-21.99	0.034	0.003
Mean	58.48	13.04	1.047
Median	73.69	5.238	0.435
Max.	97.70	83.72	6.525
Std. Dev.	36.40	19.26	1.508
Skewness	-0.906	2.184	2.104
Kurtosis	2.396	7.325	6.987

Table 8: Call option added: Summary statistics for the proximity indices, CPT versus MV

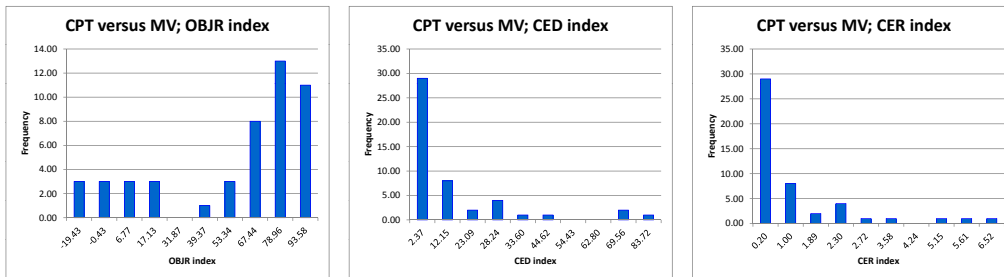


Figure 16: Call option added: Histograms for the CPT–MV comparison

Choosing again a middle range with 25 investors, regarding I_{CED} values, we get $1.9\% < I_{CED} < 25.4\%$ with the average 8.8%. This result indicates a very high deviation between the optimal portfolios obtained by direct CPT–optimization and those computed by optimization along the MV frontier.

This conclusion remains valid if we neglect the 11 investors with not PT-conform parameter values. In the middle range of the rest of investors we have $0.67\% < I_{CED} < 15.4\%$ and the average I_{CED} value is 6.1%.

One of the main reasons for this difference is the preference of CPT-investors regarding positively skewed securities, see Figure 17.

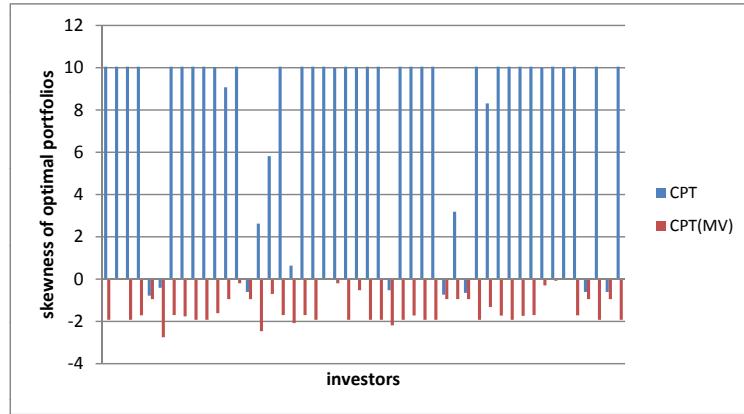


Figure 17: Call option added: The skewness of the optimal portfolios obtained via the CPT and the CPT(MV) approach.

In this case 25 investors do not diversify but invest their whole wealth into the call option and 15 further investors invest a positive fraction into that security; see Figure 37 in the Appendix. Thus our numerical results wholly reflect the preference of CPT-investors for positively skewed securities, as proved Barberis and Huang (2008) [2] under normality assumption. Notice that the role of their lottery-like security is in our case played by the European call option, having positive payoff only in a single state-of-the-world.

For checking whether the probability weighting is the single cause for this skewness-loving, we carried out a test where we have set the probability weighting parameters $\gamma = \delta = 1$. In this case CPT reduces to PT without probability weighting; the results are displayed in Figure 18.

We observe in comparison with Figure 17 that the degree of skewness-loving is reduced but it is still there. Thus, besides of probability weighting, other aspects of CPT play apparently also a role in the phenomenon of skewness-loving of CPT investors.

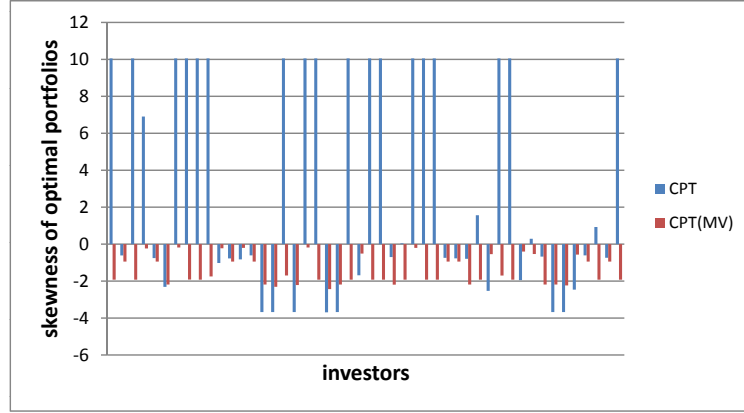


Figure 18: Call option added: The skewness of the optimal portfolios obtained via the CPT and the CPT(MV) approach, without probability weighting.

Finally we present our numerical results concerning the distances between the (σ, μ) values of the optimal CPT-portfolios and the MV-frontier. It turns out, that also with the second data-set the former do not lie along the MV frontier, see Figure 19.

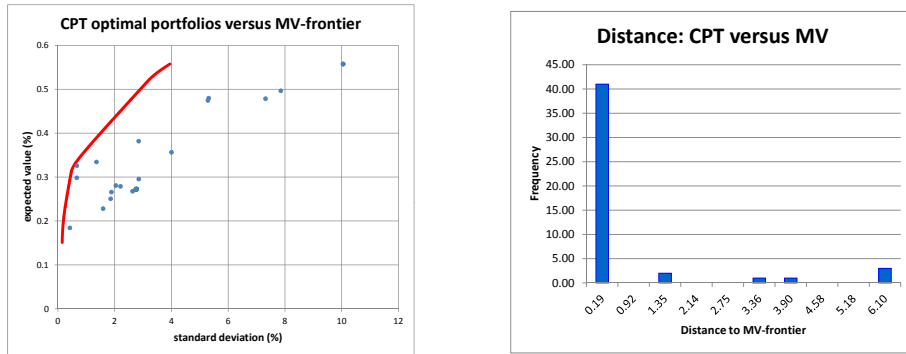


Figure 19: Call option added: Optimal CPT-portfolios versus MV frontier. On the right-hand-side the frequency-histogram of the distances to the MV-frontier is displayed.

The numerical results of checking the loss aversion parameter dependence of the distance are displayed in Figure 20 and analogously to the previous cases we see that the largest distances correspond to low β values.

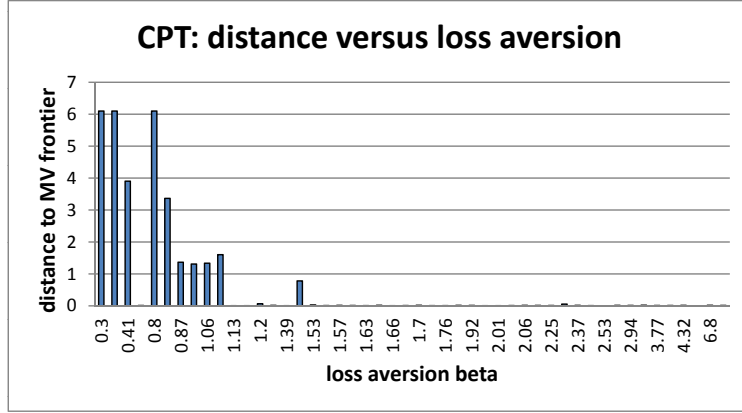


Figure 20: Call option added: The dependence of the distance to the MV-frontier on the loss-aversion parameter β , CPT case

As before, we drop the 8 portfolios corresponding to loss-aversion parameter $\beta < 1$, then we get the average distance value 0.22% which indicates that the set of optimal CPT-portfolios is not a subset of the MV efficient frontier.

Summarizing: Our computational results show that having a clear deviation from the normality assumption concerning returns, the PT and CPT portfolios may largely differ from the portfolios obtained by maximizing the PT respectively CPT objective functions along the MV-frontier. The effects discussed in the previous section are magnified now, see the results concerning the proximity indices. In particular, regarding the relation between the CPT-optimal portfolios and those obtained by maximizing the CPT objective function along the MV-frontier, we now have in annual return terms the huge average deviation $I_{CED} = 13.0\%$ with maximum value $I_{CED} = 83.7\%$, minimal value of $I_{CED} = 0.03\%$, standard deviation 19.3% and positive skewness.

We also observe a basic difference in the structure of the optimal portfolios: Most of the PT and CPT investors invest massively into the call option. In contrast to this, in the optimal portfolios along the MV-frontier the investments into the call option are negligibly small, see Figures 36–39. Thus our numerical results fully support the theoretical findings of Barberis and Huang (2008) [2] concerning positive skewness preference of CPT investors. This is despite the fact, that in our case none of the assumptions in the paper

of Barberis and Huang hold: On the one hand, the basic securities can not be considered as coming from a normal distribution and on the other hand the added lottery-like security (in our case the call option on one of the basic securities) is not stochastically independent on the basic data-set by its very nature.

Similarly as for the basic data set, we observe that the difference between PT and CPT is remarkably smaller than their difference with respect to MV.

7 Numerical results based on an associated normal distribution

In this section we present our computational results for a data-set constructed according to a normal distribution. The results are presented in an analogous way as those in the previous two sections. As before, values of all of the three proximity indices are reported in percentage terms, with values of I_{CED} representing annual returns.

	I_{OBJR}	I_{CED}	I_{CER}
Min.	87.07	-0.015	-0.001
Mean	98.75	0.160	0.013
Median	99.99	0.0004	0.00004
Max.	100.0	1.122	0.091
Std. Dev.	2.934	0.288	0.024
Skewness	-2.595	2.134	2.11
Kurtosis	8.813	6.625	6.510

Table 9: Normal approximation: Summary statistics for the proximity indices, PT versus CPT

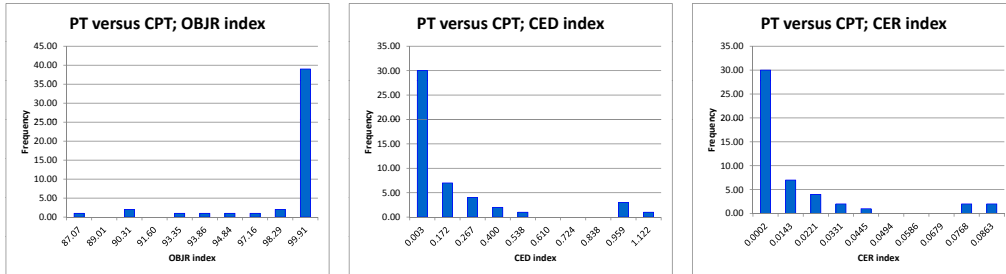


Figure 21: Normal approximation: Histograms for the PT–CPT comparison

Concerning the PT-optimization versus CPT-optimization aspect, the summary statistics of our computational results is presented in Figure 9 and Table 21 where also in this case we observe a noticeable difference.

In this case 22 investors hold identical portfolios. Reducing the considerations again to the middle range of the I_{CED} -values obtained for the rest of the investors, we find that for 14 investors we have $0.002\% < I_{CED} < 0.54\%$ with an average of 0.2% which counts as a noticeable difference, e.g. in wealth management, see De Giorgi and Hens (2009) [6].

	I_{OBJR}	I_{CED}	I_{CER}
Min.	99.98	-0.009	-0.0007
Mean	99.99	0.00002	0.000002
Median	100.0	0.000	0.000
Max.	100.0	0.005	0.0004
Std. Dev.	0.006	0.002	0.0001
Skewness	2.379	-1.895	-1.894
Kurtosis	17.13	11.73	11.72

Table 10: Normal approximation: Summary statistics for the proximity indices, PT versus MV

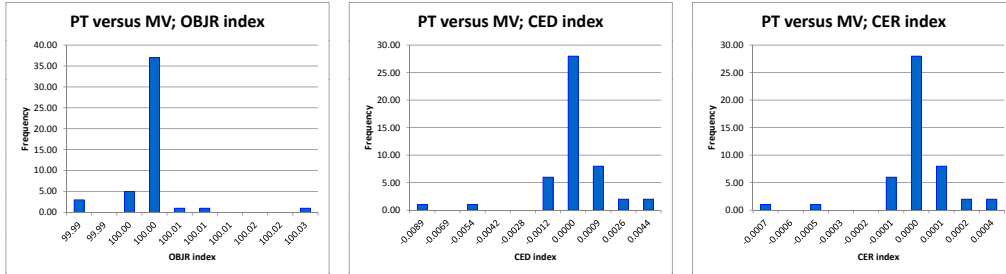


Figure 22: Normal approximation: Histograms for the PT-MV comparison

	I_{OBJR}	I_{CED}	I_{CER}
Min.	99.66	-0.001	-0.00009
Mean	99.99	0.0002	0.00001
Median	100.0	0.000	0.000
Max.	100.0	0.006	0.0005
Std. Dev.	0.049	0.001	0.00008
Skewness	-6.616	4.590	4.594
Kurtosis	44.85	28.16	28.19

Table 11: Normal approximation: Summary statistics for the proximity indices, CPT versus MV

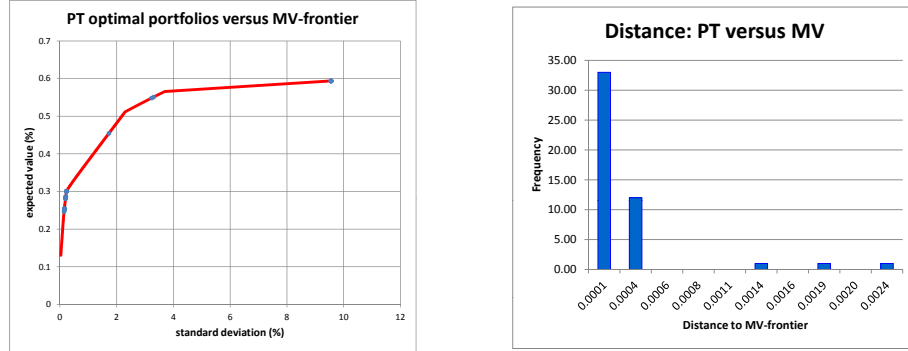


Figure 23: Normal approximation: Optimal PT-portfolios versus MV frontier and the frequency-histogram of the distances to the MV-frontier.

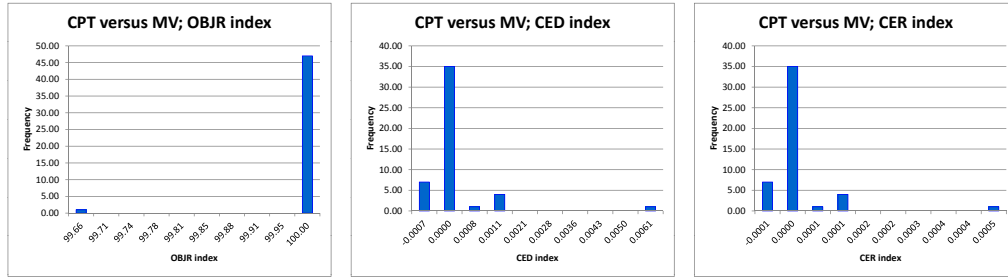


Figure 24: Normal approximation: Histograms for the CPT-MV comparison

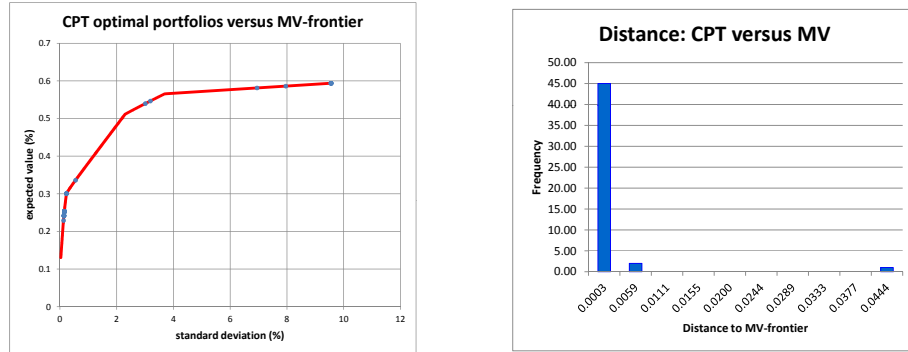


Figure 25: Normal approximation: Optimal CPT-portfolios versus MV frontier and the frequency-histogram of the distances to the MV-frontier.

Regarding the PT versus MV and the CPT versus MV comparisons, our numerical results clearly indicate that with this data-set the optimal PT and the optimal CPT portfolios are now located along the efficient frontier and this feature can be observed directly by considering the corresponding sum-

mary statistics. Therefore we do not discuss these in a detailed fashion, but just provide a list of them in the same order as in the previous sections.

Summarizing: In accordance with the theoretical result of Levy and Levy (2004) [23], both the proximity indices and the distances to the efficient MV-frontier indicate that the CPT-optimal portfolios are located now on or very close to the efficient MV-frontier. Concerning annualized returns, we now have for the average deviation of certainty equivalents $I_{CED} = 0.0002\%$, maximal value $I_{CED} = 0.006\%$, minimal value $I_{CED} = -0.001\%$, standard deviation 0.001% and again positive skewness.

Note that Levy and Levy (2004) [23] have obtained their result under allowing for unlimited short-sales whereas in our numerical investigations short-sales are forbidden. It is a remarkable fact that despite this noteworthy difference the optimal CPT-portfolios are located along the MV-frontier.

Interestingly, also the PT-optimal portfolios are located along the MV-frontier or are close to it. On the other hand, considering the proximity indices, it turns out that the optimal PT and CPT optimal portfolios differ.

Notice that, in the comparison CPT versus portfolios obtained by optimizing the CPT-objective function along the MV-frontier, also negative values occur for the I_{CED} index (for altogether 6 investors), with the minimal value of $I_{CED} = -0.001\%$, see Table 11. This means that, at least for some investors, the optimal portfolios obtained along the MV-frontier are slightly better than those obtained by CPT optimization. This is due to the fact that the optimal solution for that investors lies very close to the efficient frontier whereas the adaptive grid method delivers approximate solutions according to simplicial subdivisions of the unit simplex delivers an approximation to the solution located along the curved MV-frontier. Thus we conclude that in the case of a multivariate normal distribution the algorithm suggested by Levy and Levy [23] may deliver more accurate solutions, since it searches directly along the curved frontier.

8 Comparing the results for the three data-sets

Figures 26–31 serve for displaying the results for the three data-types in a comparative summarizing manner; for the details see the previous sections.

In these Figures the filled circles represent expected values and the upward and downward oriented vertical line-segments stand for the upper- and lower semi-deviations, respectively. The distributions of the proximity indices are clearly non-symmetric (see the previous Section) therefore we think that figures of this type provide additional intuition concerning the differences across the data-sets.

Recall that the values for all of the three indices are expressed in percentage terms. Both I_{OBJR} and I_{CER} represent ratios, whereas I_{CED} stands for differences in certainty equivalents expressed as annual returns.

The maximal value for I_{OBJR} is 100%, with larger values representing larger proximity. Contrary to this, for the other two indices I_{CED} and I_{CER} , larger values correspond to larger dissimilarity between the evaluated portfolios.

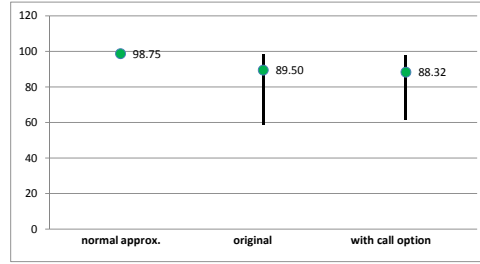


Figure 26: PT versus CPT; the index I_{OBJR} across the three data-sets

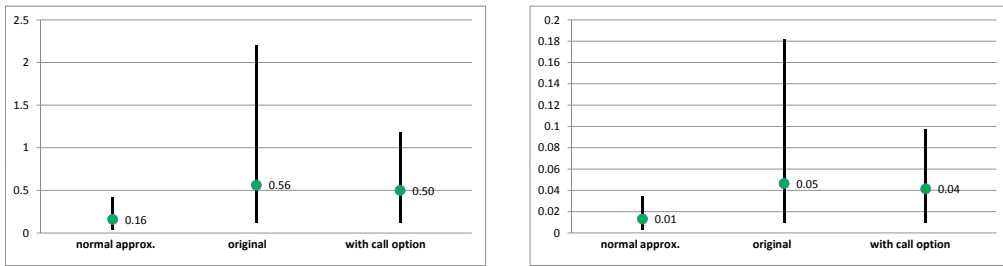


Figure 27: PT versus CPT; the indices I_{CED} and I_{CER} across the three data-sets

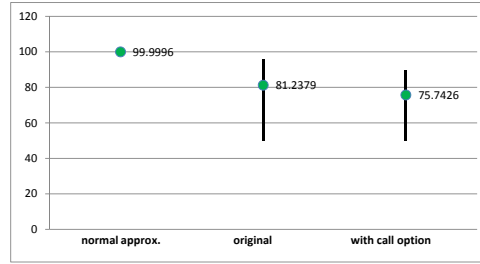


Figure 28: PT versus MV; index I_{OBJR} across the three data-sets

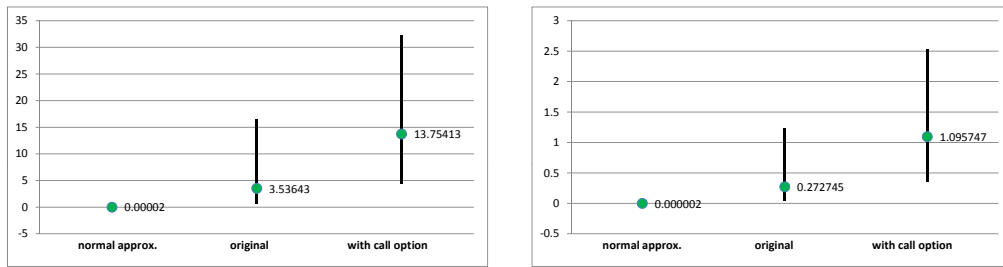


Figure 29: PT versus MV; indices I_{CED} and I_{CER} across the three data-sets

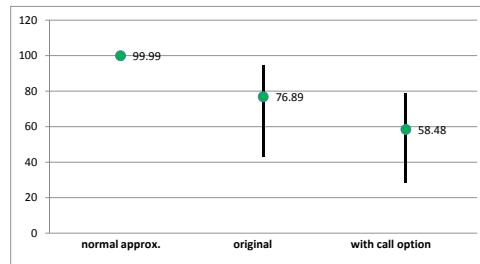


Figure 30: CPT versus MV; index I_{OBJR} across the three data-sets

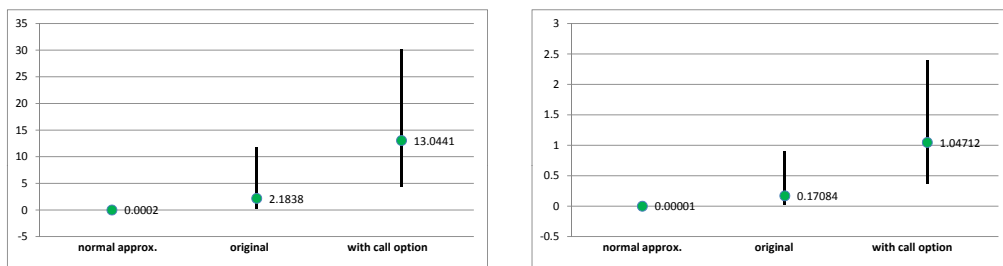


Figure 31: CPT versus MV; indices I_{CED} and I_{CER} across the three data-sets

9 Conclusion

To the best of our knowledge our paper proposes the first general algorithm for computing asset allocations for cumulative prospect theory. This is a numerically hard problem that is of high relevance for finance. Based on the implementation of the algorithm, we have carried out a numerical study based on a real-life data-set and two variants of it, with the goal of numerically testing the differences between the two types of prospect theory models and between the prospect theory approach and mean-variance analysis.

We find that asset allocations based on cumulative prospect theory are different from those coming from prospect theory, the difference being the smallest when the return distribution is normal. Except for normally distributed returns, we observe that both prospect theory models differ substantially from mean-variance analysis. As one of the reasons we identify the preference of CPT-investors for positively skewed securities and numerically verify this by adding a call option to our original data set.

Further research should compare the three asset allocation models considered there “out of sample”. In particular it is then interesting whether the simple $1/n$ rule that beats the mean-variance model out of sample can also beat asset allocations based on prospect theory.

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10 Appendix: Tables

	GSCITR	HFRIFFM	I3M	JPMBD	MSEM	MXWO	NAREIT	PE
Min.	-3.7205	-2.3395	-0.0885	-0.4539	-5.1791	-2.1162	-3.0722	-3.1710
Mean	0.0441	0.0584	0.0149	0.0364	0.0669	0.0669	0.0546	0.0594
Median	0.0909	0.1208	0.0065	0.0303	0.0872	0.1259	0.1402	0.1283
Max.	2.3879	1.3723	0.1839	0.5459	3.8823	1.6683	2.2549	3.2807
Std. Dev.	0.8279	0.4503	0.0762	0.1732	1.2644	0.5335	0.6409	0.7517
Skewness	-0.4903	-1.8369	0.4415	0.0302	-0.5283	-0.6643	-0.8683	-0.2963
Kurtosis	4.5874	10.604	2.1578	3.5438	4.7248	4.8595	7.6889	8.0357

Table 12: Summary statistics for the monthly-returns data-set. The statistics is for annualized data, for facilitating comparisons based on the I_{CED} proximity index.

	GSCITR	HFRIFFM	I3M	JPMBD	MSEM	MXWO	NAREIT	PE
GSCITR	1							
HFRIFFM	0.4212	1						
I3M	-0.0666	-0.1257	1					
JPMBD	0.1444	-0.0533	-0.0324	1				
MSEM	0.3693	0.57204	-0.2365	0.0579	1			
MXWO	0.3687	0.5414	-0.1788	0.1676	0.8122	1		
NAREIT	0.3636	0.4052	-0.1960	0.2667	0.7254	0.8045	1	
PE	0.4186	0.5671	-0.1939	0.1348	0.7298	0.8306	0.7273	1

Table 13: Empirical correlation matrix of the monthly-returns data-set

	Original Data-Set N=208	Normal sample N=208	Normal sample N=10'000
mv. skewness	15.88	3.08	0.06
test stat. $\chi^2(120)$	550.62	106.65	104.14
p -value	0.000	0.80	0.85
mv. kurtosis	113.12	77.57	79.74
test stat. $\mathcal{N}(0, 1)$	18.88	-1.38	-1.02
p -value	0.000	0.17	0.31

Table 14: Mardia's tests for multivariate normality; observed values of test statistics for multivariate skewness and kurtosis along with the corresponding p -values. The results concerning samples from the multivariate normal distribution are presented for comparative purposes.

P	GSCITR	HFRIFFM	I3M	JPMBD	MSEM	MXWO	NAREIT	PE	Call on MXWO
0.1731	0.0441	0.0168	0.0012	0.0060	0.0484	0.0268	0.0315	0.0191	0
0.1635	-0.0024	0.0041	0.0023	0.0062	-0.0157	0.0023	0.0107	0.0109	0
0.1490	-0.0384	0.0213	0.0002	-0.0012	0.0938	0.0318	0.0205	0.0271	0
0.1346	0.0756	0.0098	0.0009	0.0030	-0.0175	-0.0095	-0.0213	-0.0044	0
0.1010	-0.0816	-0.0025	0.0007	-0.0023	-0.0294	-0.0082	-0.0052	-0.0174	0
0.0865	0.0654	0.0350	-0.0023	0.0056	0.1353	0.0500	0.04918	0.0682	0
0.0865	0.0014	-0.0261	0.0060	0.0008	-0.1334	-0.0518	-0.0401	-0.0691	0
0.0385	-0.0284	-0.0117	0.0051	0.0023	-0.1663	-0.0074	-0.0354	0.0013	0
0.0241	-0.1388	-0.0376	-0.0029	-0.0038	-0.1643	-0.1059	-0.1447	-0.1658	0
0.0096	0.0370	0.0634	-0.0019	-0.0024	0.1929	0.0840	0.0978	0.2623	0
0.0096	-0.0574	-0.0743	-0.0026	0.0266	0.1505	0.0933	0.1499	0.0562	0
0.0096	0.1482	0.0240	-0.0058	0.0168	0.3165	0.1232	0.1467	0.1585	0.2832
0.0048	-0.3100	-0.1935	0.0064	-0.0113	-0.3817	-0.1763	-0.2560	-0.2643	0
0.0048	-0.1459	-0.1950	0.0080	-0.0035	-0.2370	-0.1050	-0.0862	-0.1193	0
0.0048	-0.0314	-0.1313	0.0005	0.0252	-0.4316	-0.1259	-0.0833	-0.1161	0

Table 15: The empirical distribution, constructed via k -means clustering ($k = 15$) from the monthly returns data-set with the last column corresponding to the appended European call-option on MXWO.

11 Appendix: Figures

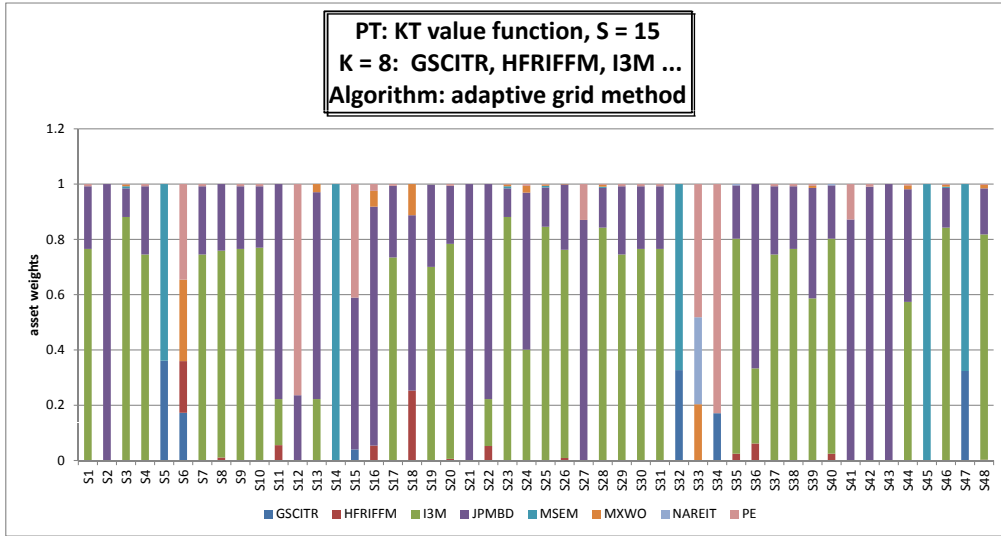


Figure 32: Maximizing PT with the piecewise-power value function

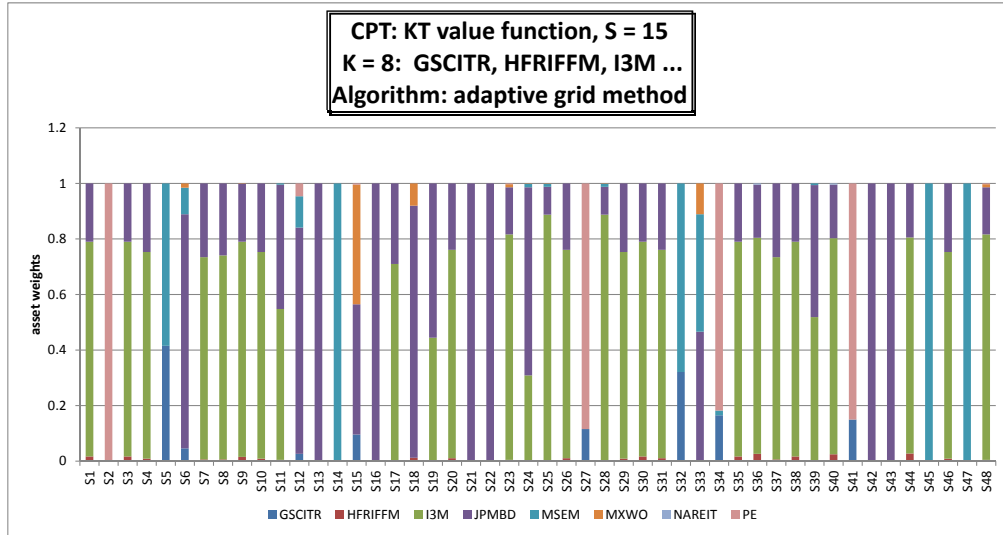


Figure 33: Maximizing CPT with the piecewise-power value function

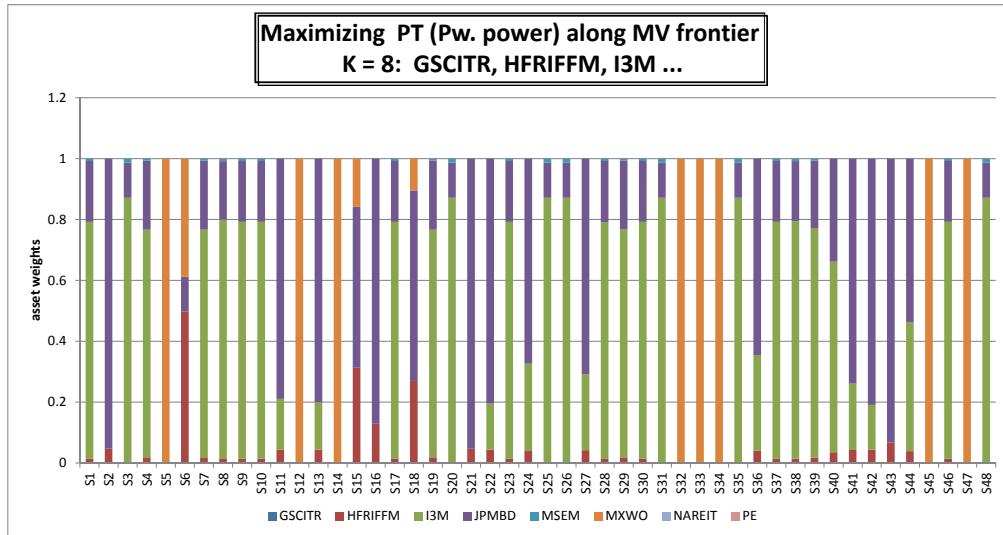


Figure 34: Maximizing PT with the piecewise-power value function, along the MV-frontier

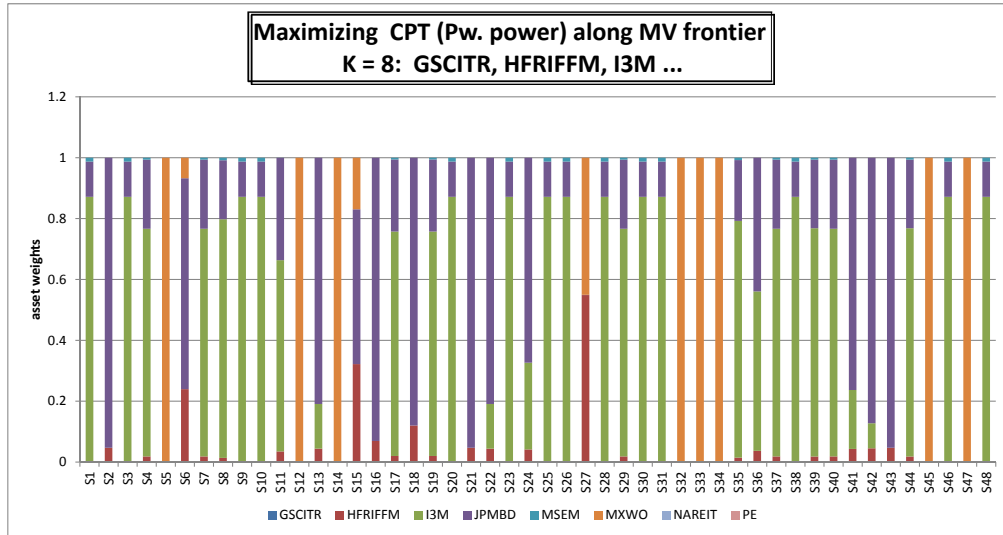


Figure 35: Maximizing CPT with the piecewise–power value function, along the MV–frontier

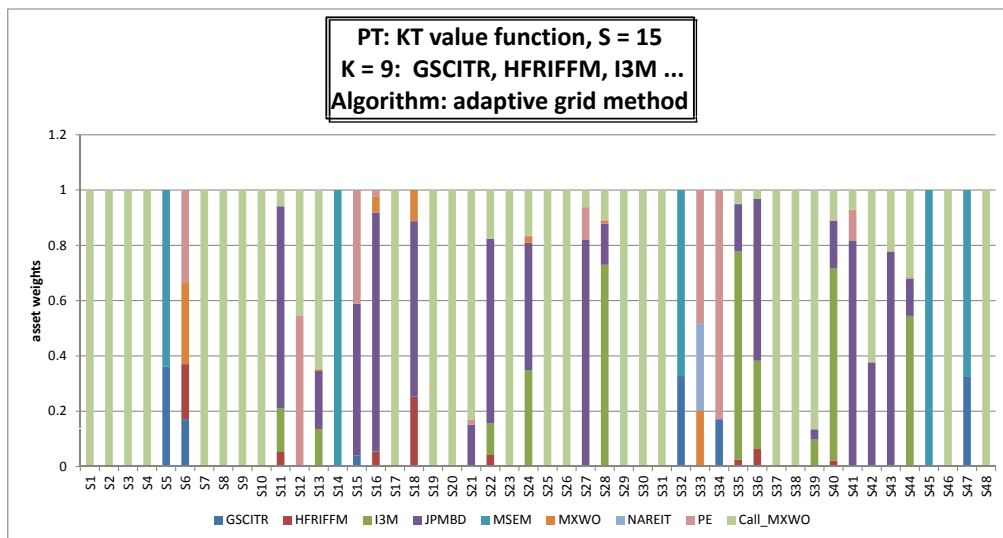


Figure 36: Call option added: Maximizing PT with the piecewise–power value function

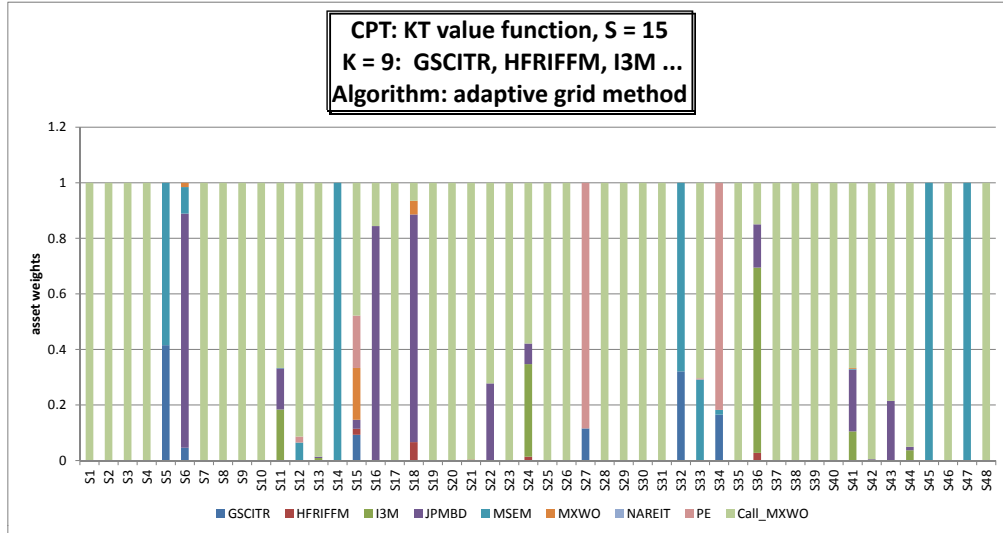


Figure 37: Call option added: Maximizing CPT with the piecewise-power value function

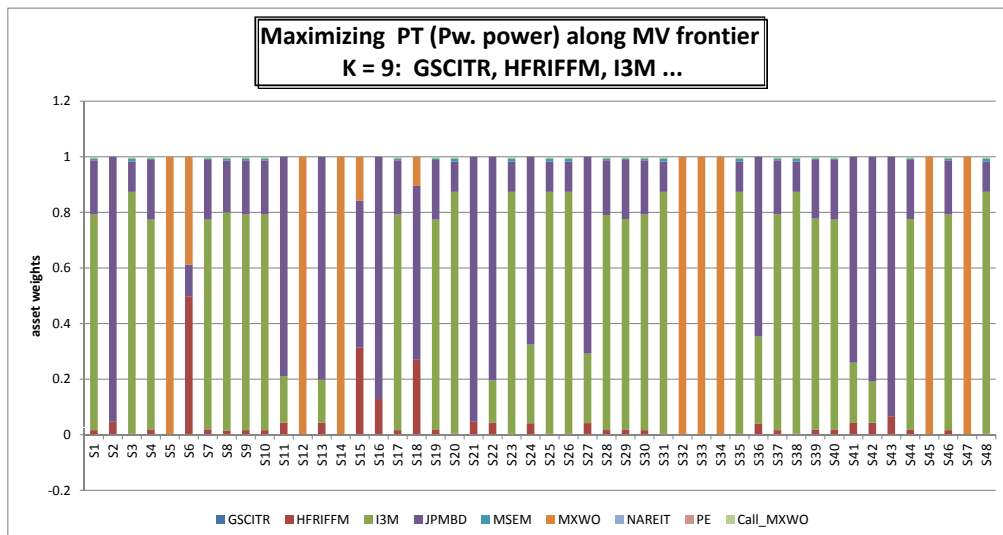


Figure 38: Call option added: Maximizing PT with the piecewise-power value function, along the MV-frontier

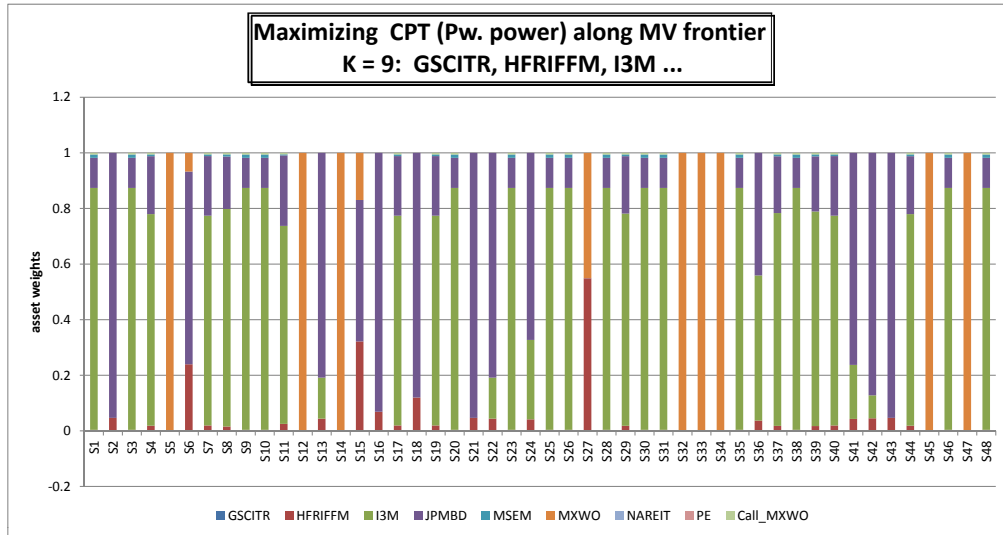


Figure 39: Call option added: Maximizing CPT with the piecewise-power value function, along the MV-frontier

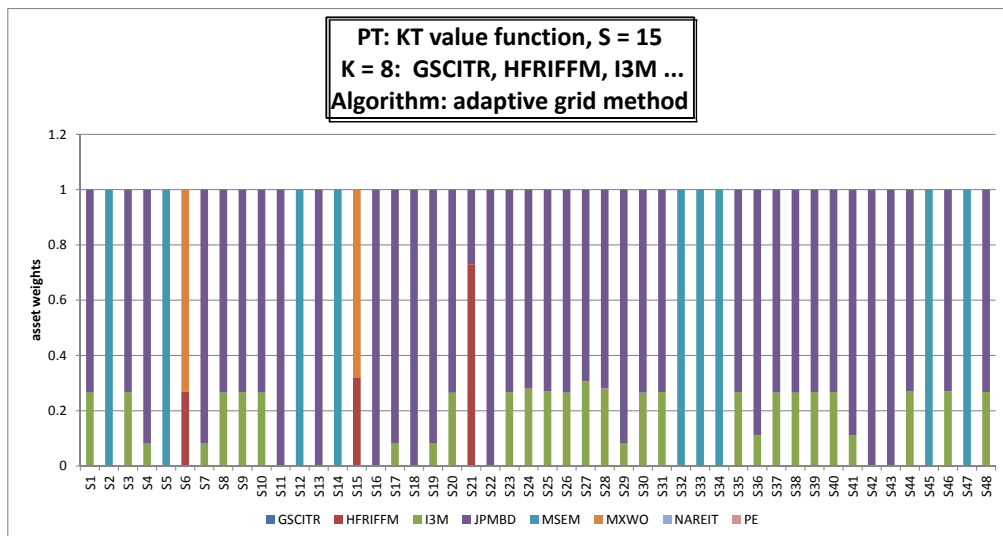


Figure 40: Normal approximation: Maximizing PT with the piecewise-power value function

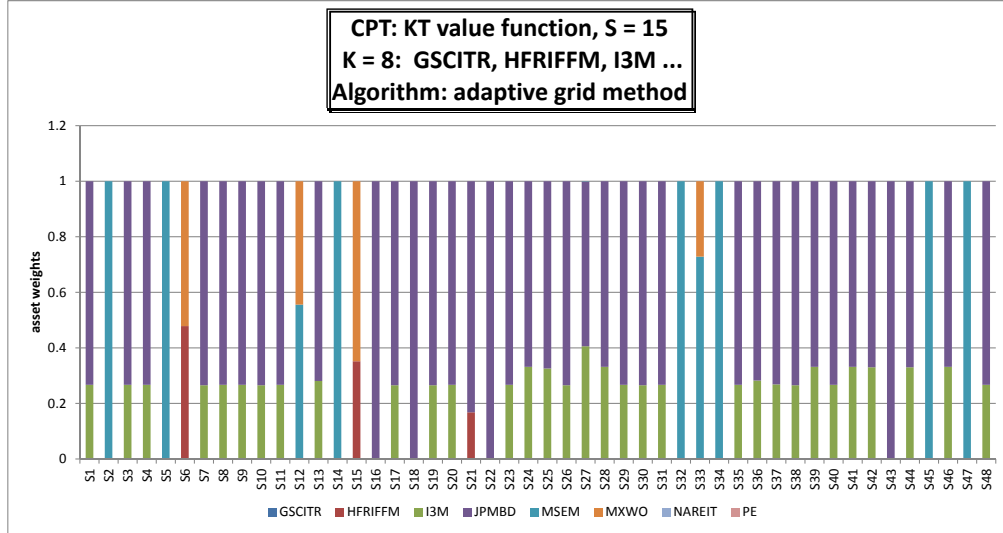


Figure 41: Normal approximation: Maximizing CPT with the piecewise-power value function

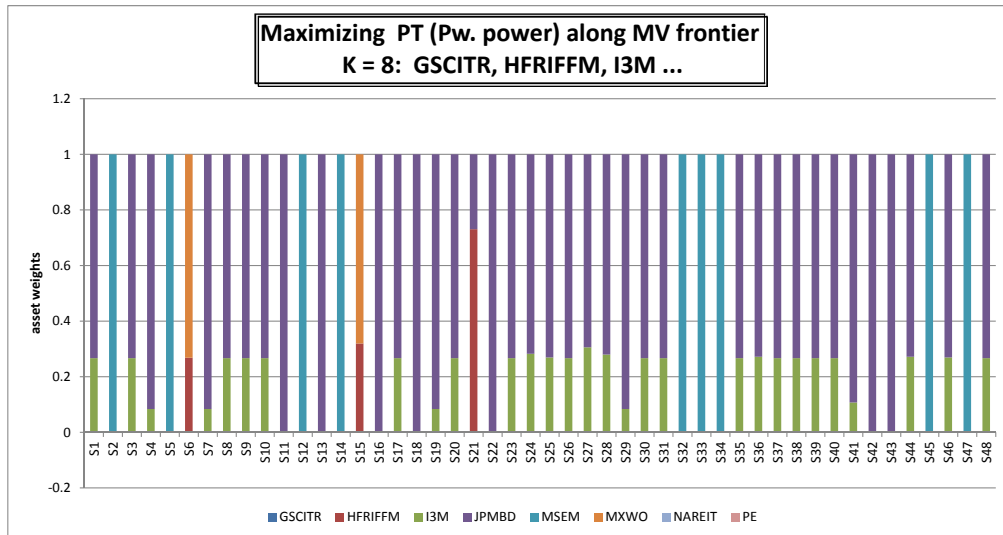


Figure 42: Normal approximation: Maximizing PT with the piecewise-power value function, along the MV-frontier

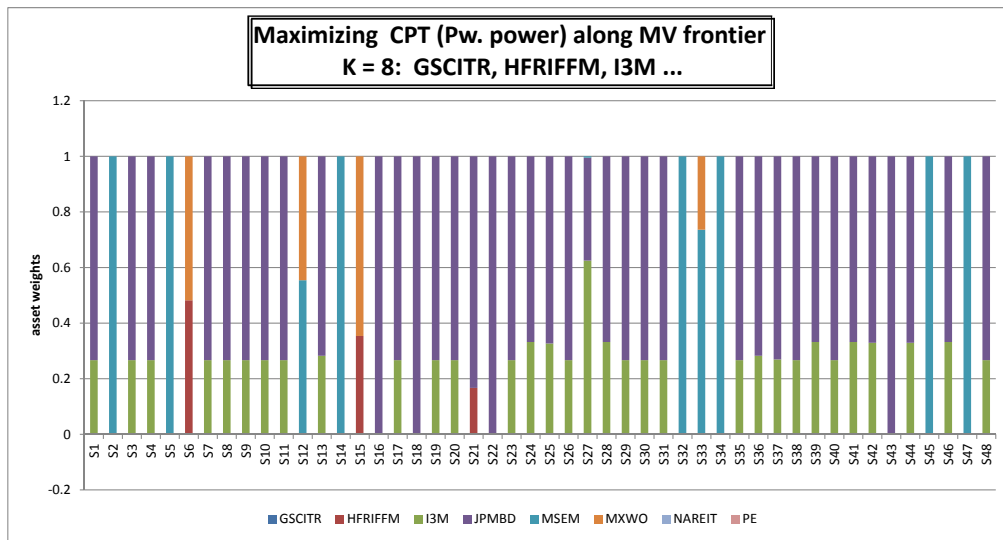


Figure 43: Normal approximation: Maximizing CPT with the piecewise-power value function, along the MV-frontier